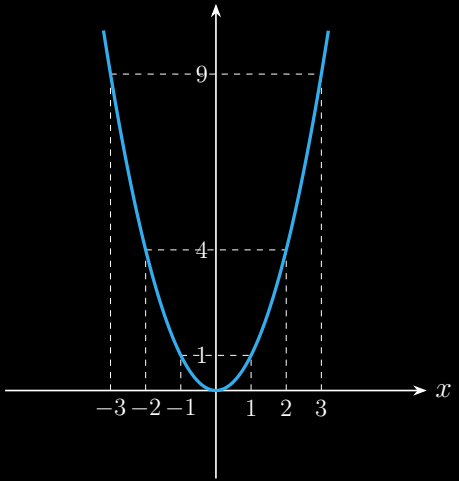
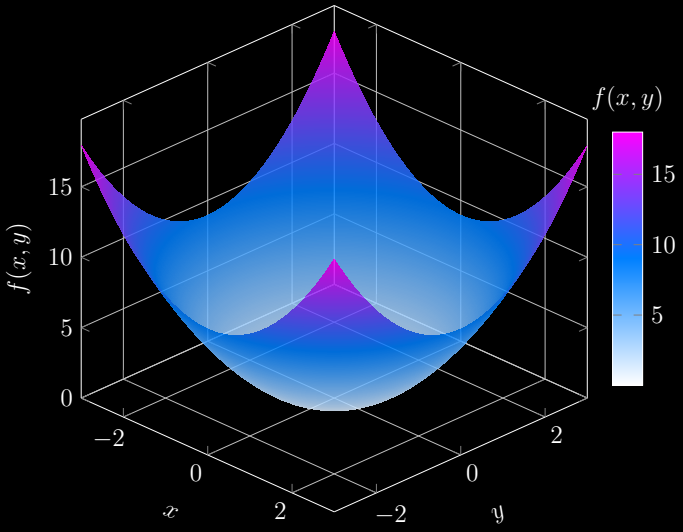
Quadratic Function

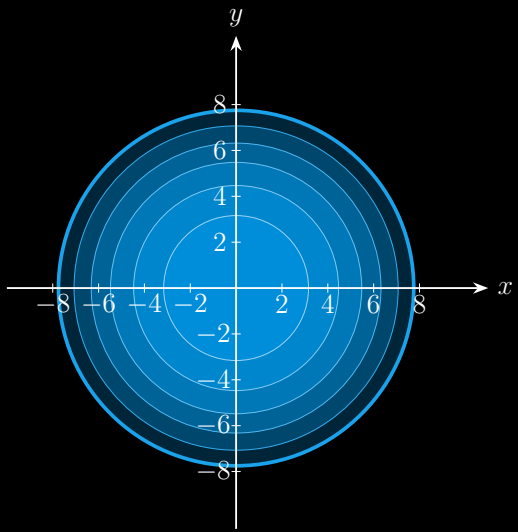
$$f(x) = x^2$$



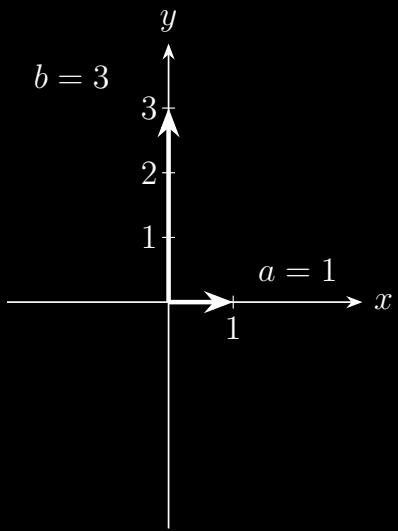
Quadratic Function $f(x, y) = x^2 + y^2$

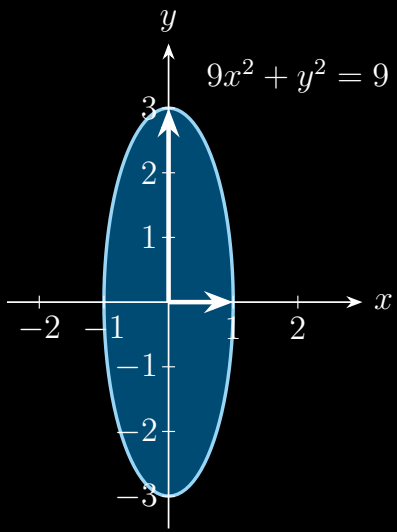


Quadratic Function $f(x, y) = x^2 + y^2$

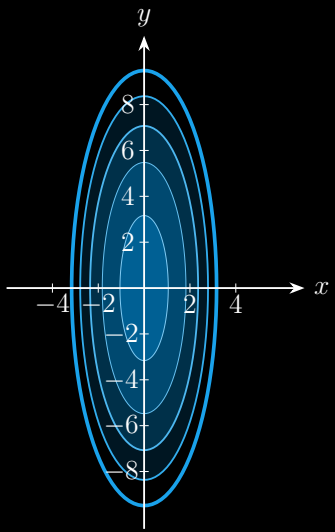


Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

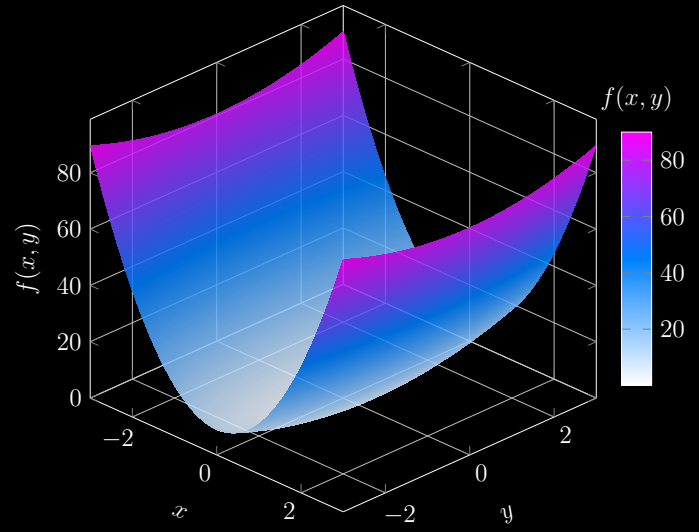

y
b = 3
3
2
1
a = 1
x
1


y
9x^2 + y^2 = 9
3
2
1
−1
−2
−3
x
−2
−1
1
2

Quadratic Function $f(x, y) = 9x^2 + y^2$



Quadratic Function $f(x, y) = 9x^2 + y^2$



Quadratic Function

$$f(x,y) = 9x^2 + 4xy + y^2 = \begin{bmatrix} x \\ y \end{bmatrix}^\top \underbrace{\begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvalue decomposition:

$$\boldsymbol{A} = \lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^\top + \lambda_2 \boldsymbol{u}_2 \boldsymbol{u}_2^\top$$

with eigenvalues:

$$\lambda_1 = 9.47 \quad \lambda_2 = 0.53$$

and eigenvectors:

$$\boldsymbol{u}_1 = \begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} -0.23 \\ 0.97 \end{bmatrix}$$

Quadratic Function

$$f(x,y) = 9x^2 + 4xy + y^2 = \begin{bmatrix} x \\ y \end{bmatrix}^\top \underbrace{\begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvalue decomposition:

$$\boldsymbol{A} = \lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^\top + \lambda_2 \boldsymbol{u}_2 \boldsymbol{u}_2^\top$$

with eigenvalues:

$$\lambda_1 = 9.47 \quad \lambda_2 = 0.53$$

and eigenvectors:

$$\boldsymbol{u}_1 = \begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} -0.23 \\ 0.97 \end{bmatrix}$$

```python
import numpy as np

A = np.array([[9, 2], [2, 1]])
eig_val, eig_vec = np.linalg.eig(A)

print('lambda_1 = ', eig_val[0])
print('lambda_2 = ', eig_val[1])
print('u_1 = ', eig_vec[:, 0])
print('u_2 = ', eig_vec[:, 1])
```

Quadratic Function

$$f(x,y) = 9x^2 + 4xy + y^2 = \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}^\top}_{\boldsymbol{w}^\top} \underbrace{\begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{w}}$$

Using the eigenvalue decomposition:

$$\begin{aligned} \boldsymbol{w}^\top \boldsymbol{A} \boldsymbol{w} &= \boldsymbol{w}^\top \left(\lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^\top + \lambda_2 \boldsymbol{u}_2 \boldsymbol{u}_2^\top \right) \boldsymbol{w} \\ &= \lambda_1 (\boldsymbol{u}_1^\top \boldsymbol{w})^2 + \lambda_2 (\boldsymbol{u}_2^\top \boldsymbol{w})^2 \end{aligned}$$

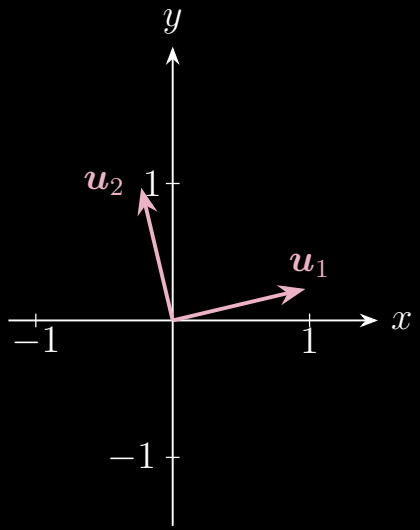
with eigenvalues and eigenvectors:

$$\lambda_1 = 9.47 \quad \boldsymbol{u}_1 = \begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix} \quad \lambda_2 = 0.53 \quad \boldsymbol{u}_2 = \begin{bmatrix} -0.23 \\ 0.97 \end{bmatrix}$$

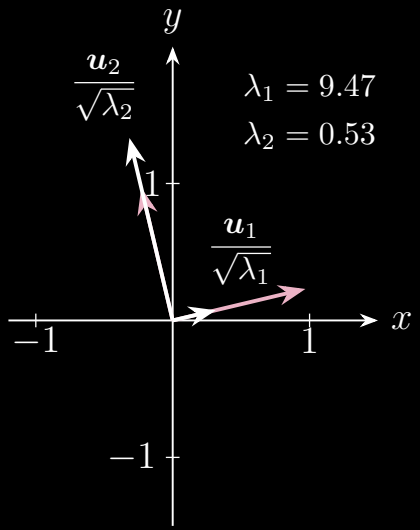
If $f(x, y) = 1$, then the ellipse is

$$\lambda_1(\boldsymbol{u}_1^\top \boldsymbol{w})^2 + \lambda_2(\boldsymbol{u}_2^\top \boldsymbol{w})^2 = 1 \quad \Rightarrow \quad \frac{(\boldsymbol{u}_1^\top \boldsymbol{w})^2}{\left(1/\sqrt{\lambda_1}\right)^2} + \frac{(\boldsymbol{u}_2^\top \boldsymbol{w})^2}{(1/\sqrt{\lambda_2})^2} = 1$$

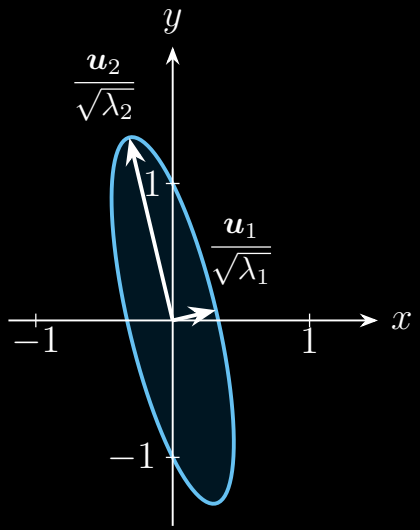
Ellipse $\frac{(\boldsymbol{u}_1^\top \boldsymbol{w})^2}{(1/\sqrt{\lambda_1})^2} + \frac{(\boldsymbol{u}_2^\top \boldsymbol{w})^2}{(1/\sqrt{\lambda_2})^2} = 1$



Ellipse $\frac{(\boldsymbol{u}_1^\top \boldsymbol{w})^2}{(1/\sqrt{\lambda_1})^2} + \frac{(\boldsymbol{u}_2^\top \boldsymbol{w})^2}{(1/\sqrt{\lambda_2})^2} = 1$



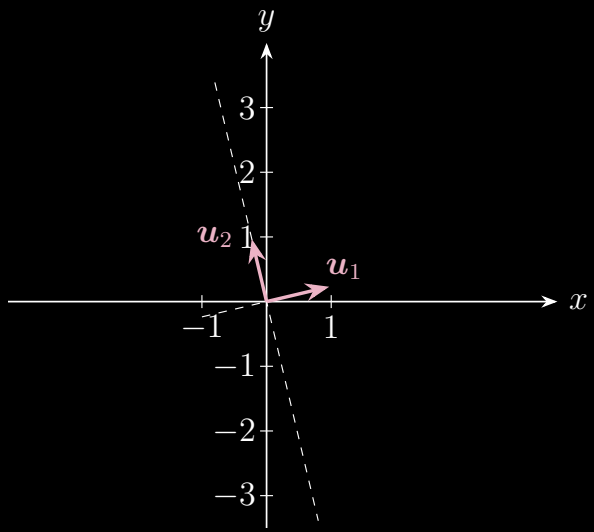
Ellipse $\frac{(\boldsymbol{u}_1^\top \boldsymbol{w})^2}{(1/\sqrt{\lambda_1})^2} + \frac{(\boldsymbol{u}_2^\top \boldsymbol{w})^2}{(1/\sqrt{\lambda_2})^2} = 1$



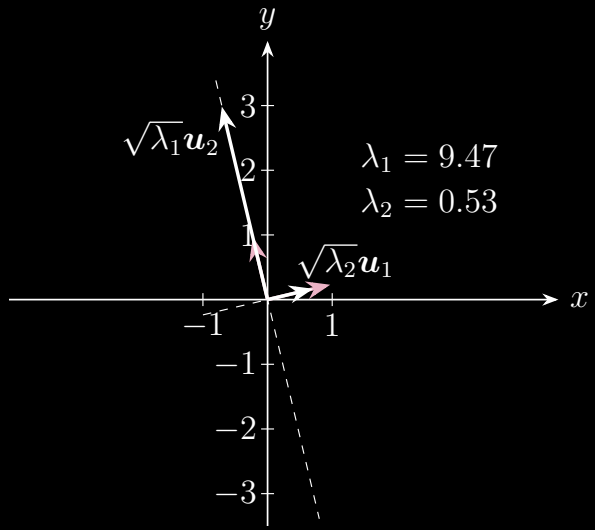
If $f(x, y) = \lambda_1 \lambda_2$, then the ellipse is

$$\lambda_1 (\boldsymbol{u}_1^\top \boldsymbol{w})^2 + \lambda_2 (\boldsymbol{u}_2^\top \boldsymbol{w})^2 = \lambda_1 \lambda_2 \quad \Rightarrow \quad \frac{(\boldsymbol{u}_1^\top \boldsymbol{w})^2}{(\sqrt{\lambda_2})^2} + \frac{(\boldsymbol{u}_2^\top \boldsymbol{w})^2}{(\sqrt{\lambda_1})^2} = 1$$

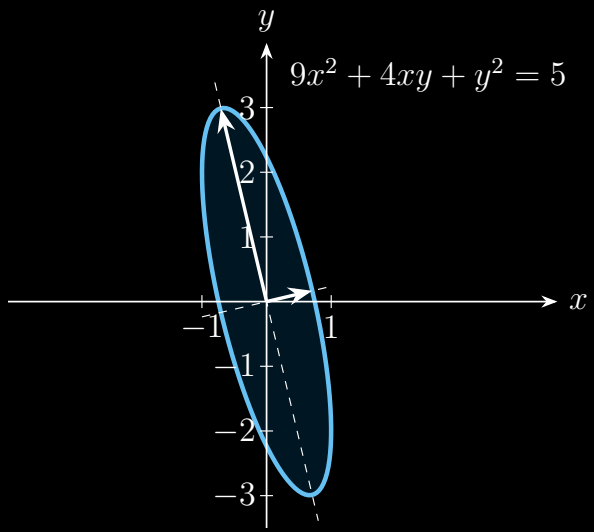
Ellipse $\frac{(\boldsymbol{u}_1^\top \boldsymbol{w})^2}{(\sqrt{\lambda_2})^2} + \frac{(\boldsymbol{u}_2^\top \boldsymbol{w})^2}{(\sqrt{\lambda_1})^2} = 1$



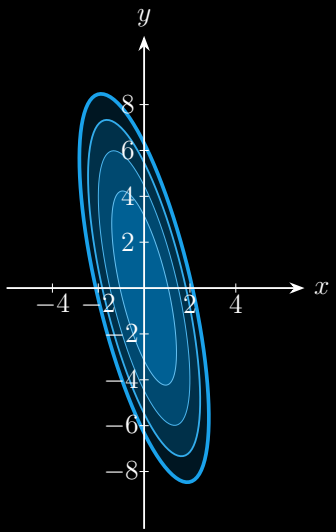
Ellipse $\frac{(\boldsymbol{u}_1^\top \boldsymbol{w})^2}{(\sqrt{\lambda_2})^2} + \frac{(\boldsymbol{u}_2^\top \boldsymbol{w})^2}{(\sqrt{\lambda_1})^2} = 1$



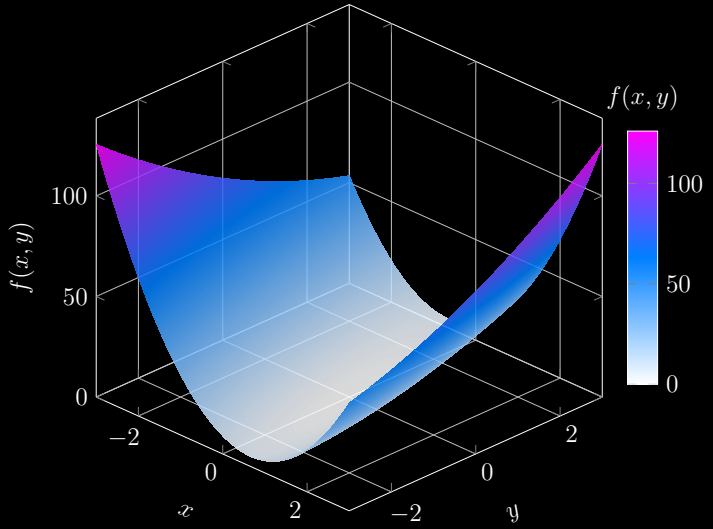
Ellipse $\frac{(\boldsymbol{u}_1^\top \boldsymbol{w})^2}{(\sqrt{\lambda_2})^2} + \frac{(\boldsymbol{u}_2^\top \boldsymbol{w})^2}{(\sqrt{\lambda_1})^2} = 1$



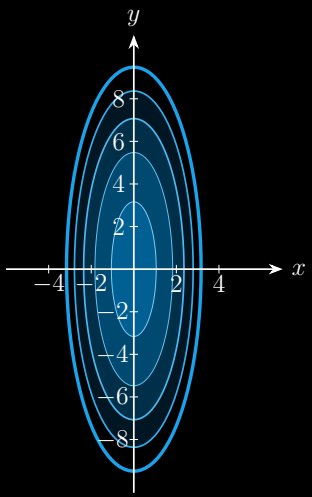
Quadratic Function $f(x, y) = 9x^2 + 4xy + y^2$

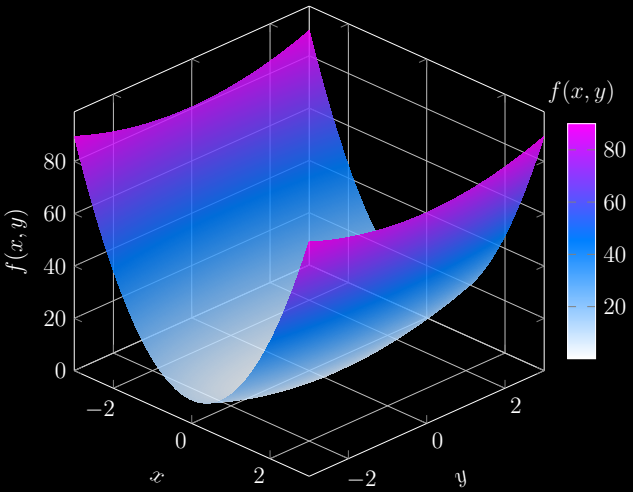


Quadratic Function $f(x, y) = 9x^2 + 4xy + y^2$

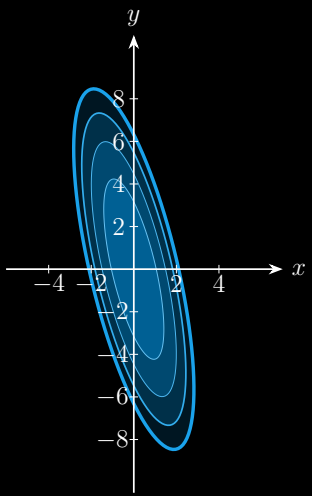


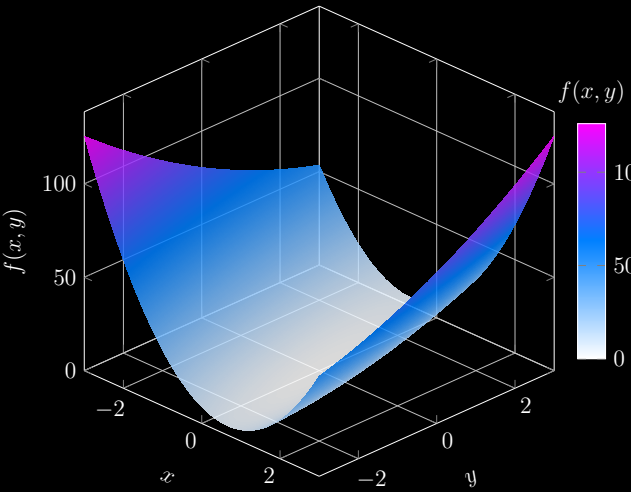
The Geometry of Quadratic Function $f(x, y) = 9x^2 + y^2$





The Geometry of Quadratic Function $f(x, y) = 9x^2 + 4xy + y^2$





Thanks for your attention!

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