Piecewise Linear Regression

Convex optimization problem:

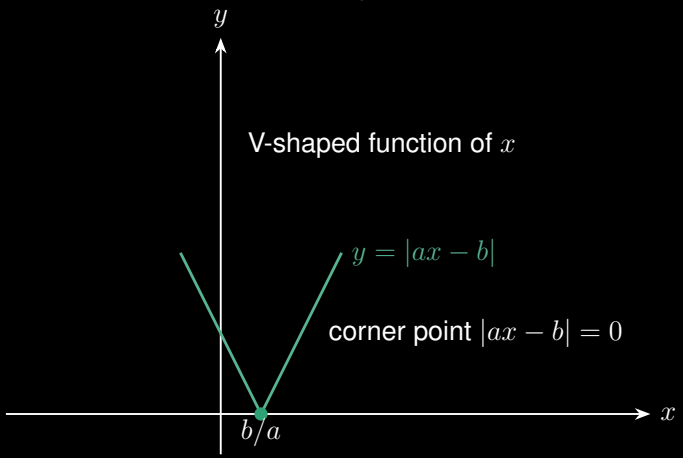
$$\min_{x} \quad f(x) = \sum_{i=1}^{n} |a_i x - b_i|$$

with data pairs $\{(a_i, b_i)\}_{i=1}^{n}$.

How to minimize $f(x)$ with a simple method?

Piecewise Linear Regression

$$\min_{x} \quad f(x) = \sum_{i=1}^{n} |a_i x - b_i|$$



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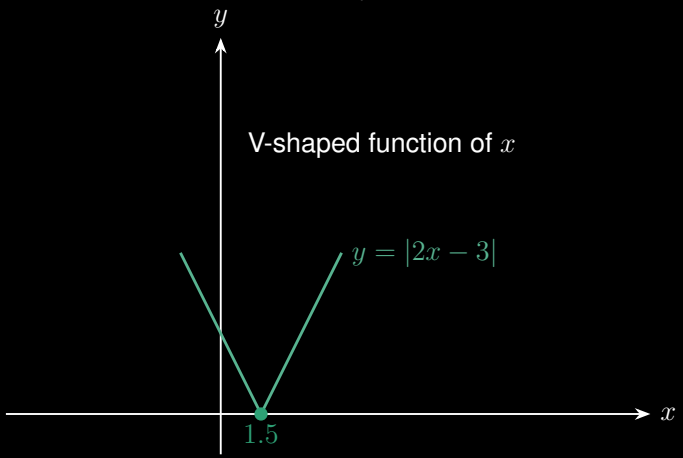

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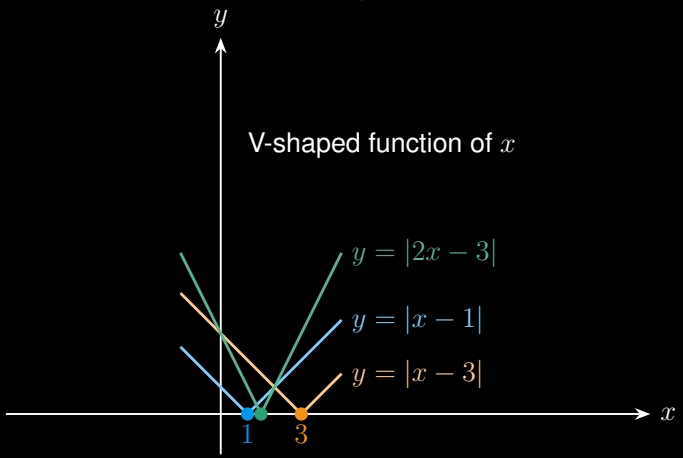

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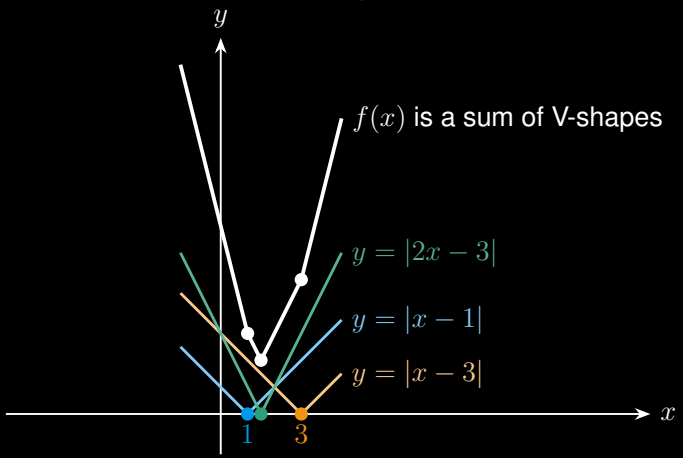

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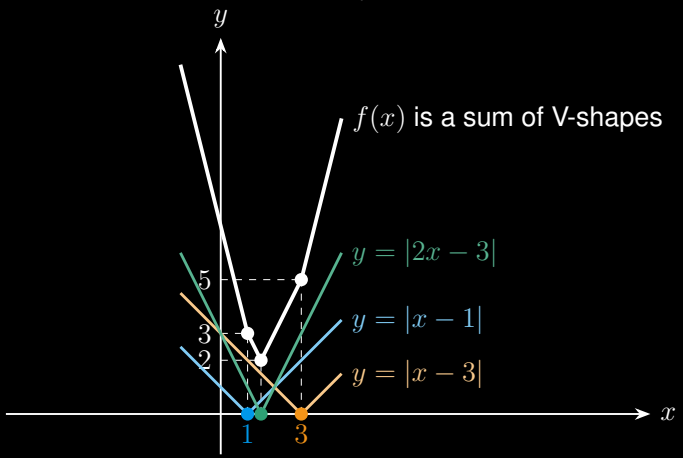

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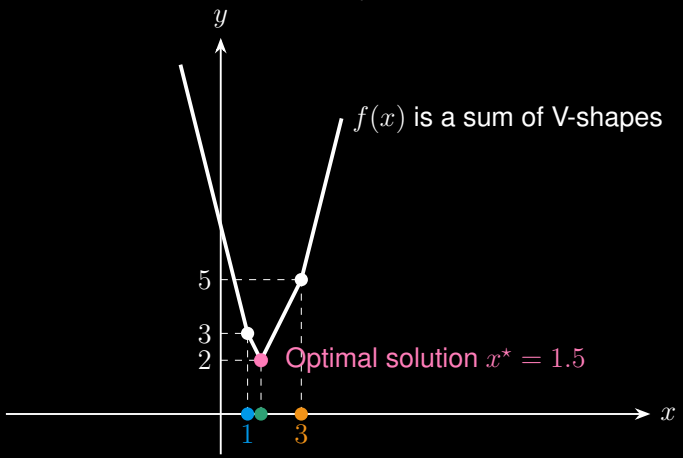
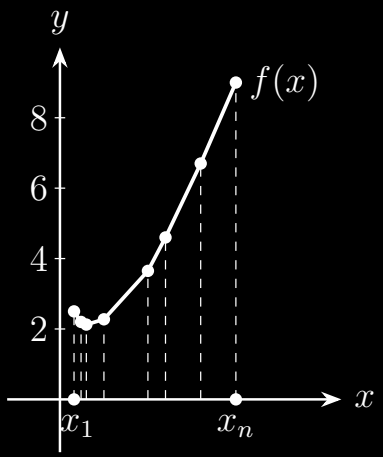


Piecewise Linear Regression

$$\min_{x} \quad f(x) = \sum_{i=1}^{n} |a_i x - b_i|$$

a_i	b_i	Corner point $x_i = b_i/a_i$
0.4	0.5	1.25
0.5	0.2	0.4
0.1	0.3	3
0.2	0.5	2.5
0.4	0.3	0.75
0.1	0.5	5
0.1	0.4	4
0.5	0.3	0.6

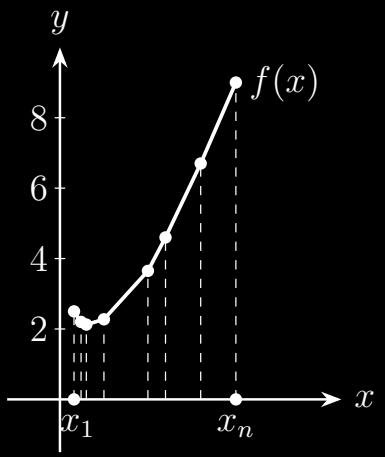


Piecewise Linear Regression

$$\min_{x} \quad f(x) = \sum_{i=1}^{n} |a_i x - b_i|$$

Sort values $x_1 \leq x_2 \leq \ldots \leq x_n$:

$x_i \in \{0.4,\ 0.6,\ 0.75,\ 1.25,\ 2.5,\ 3,\ 4,\ 5\}$



Piecewise Linear Regression (Binary Search)

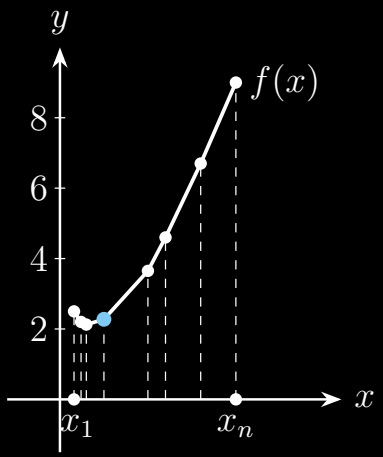
$$\min_{x} \quad f(x) = \sum_{i=1}^{n} |a_i x - b_i|$$

Compute $k = \lfloor \frac{n}{2} \rfloor = 4$:

$$x_i \in \{0.4,\ 0.6,\ 0.75,\ \underbrace{1.25}_{x_4},\ 2.5,\ 3,\ 4,\ 5\}$$

- Positive slope on $[x_4, x_5]$
- Positive slope on $[x_3, x_4]$

$f(x_4)$ is not the minimum value.



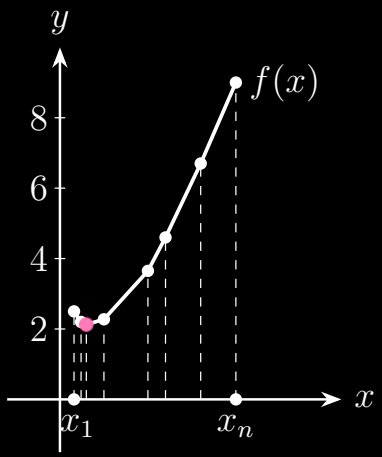
Piecewise Linear Regression (Binary Search)

$$\min_x \quad f(x) = \sum_{i=1}^{n} |a_i x - b_i| \quad \text{where the optimal solution is} \quad x^\star = 0.75$$

$$x_i \in \{0.4,\ 0.6,\ \underbrace{0.75}_{x_3},\ 1.25,\ 2.5,\ 3,\ 4,\ 5\}$$

- Positive slope on $[x_3, x_4]$
- Negative slope on $[x_2, x_3]$

$f(x_3)$ is the minimum value.



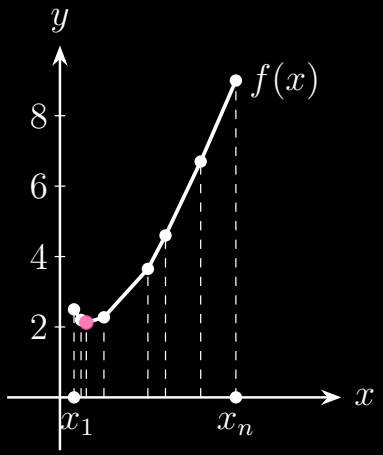
Piecewise Linear Regression

$$\min_{x} \quad f(x) = \sum_{i=1}^{n} |a_i x - b_i|$$

Algorithm:

① Compute all points $x_i = b_i/a_i$;

② Sort values $x_1 \leq x_2 \leq \ldots \leq x_n$;



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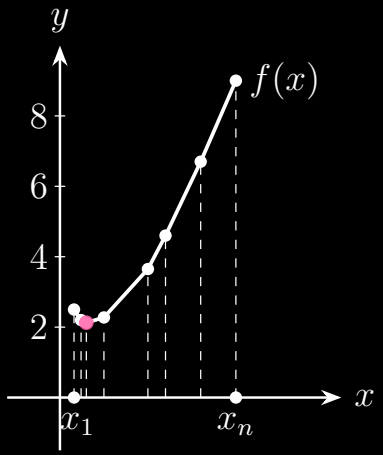
Algorithm:

① Compute all points $x_i = b_i/a_i$;

② Sort values $x_1 \le x_2 \le \ldots \le x_n$;

③ Let $k = \lfloor \frac{n}{2} \rfloor$ and compute the slope

$$s = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$



Piecewise Linear Regression

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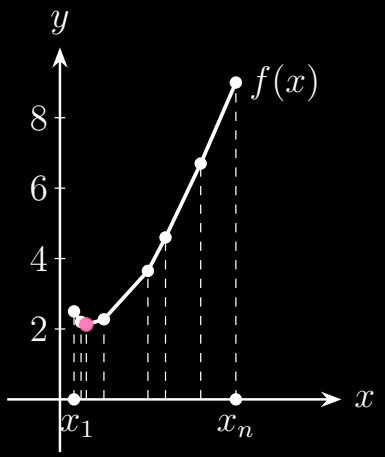
② Sort values $x_1 \le x_2 \le \ldots \le x_n$;

③ Let $k = \lfloor \frac{n}{2} \rfloor$ and compute the slope

$$s = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$

④ Find some x_ℓ such that the slope is either zero on $[x_\ell, x_{\ell+1}]$ or negative on $[x_{\ell-1}, x_\ell]$ but positive on $[x_\ell, x_{\ell+1}]$.



Piecewise Linear Regression with Linear Programming

$$\min_{x} \quad f(x) = \sum_{i=1}^{n} |a_i x - b_i|$$

Linear programming problem:

$$\min_{x, t_1, \ldots, t_n} \quad \sum_{i=1}^{n} t_i$$

$$\text{s.t.} \quad -t_i \leq a_i x - b_i \leq t_i$$

CVXPY: Convex optimization solver in Python.

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CVXPY: Convex optimization solver in Python.

```python
import cvxpy as cp

def piece_lr(a, b, n):
    x = cp.Variable(1)
    t = cp.Variable(n)

    obj = cp.Minimize(cp.sum(t))
    res = a * x - b
    constraints = [-t <= res, res <= t]
    prob = cp.Problem(obj, constraints)
    result = prob.solve()

    return x.value
```

Piecewise Linear Regression with Linear Programming

$$\min_{x} \quad f(x) = \sum_{i=1}^{n} |a_i x - b_i| \quad \text{where the optimal solution is} \quad x^\star = 0.75$$

a_i	b_i
0.4	0.5
0.5	0.2
0.1	0.3
0.2	0.5
0.4	0.3
0.1	0.5
0.1	0.4
0.5	0.3

```python
import numpy as np

n = 8
a = np.array([0.4, 0.5, 0.1, 0.2, 0.4, 0.1,
    0.1, 0.5])
b = np.array([0.5, 0.2, 0.3, 0.5, 0.3, 0.5,
    0.4, 0.3])

x = piece_lr(a, b, n)
print(x)
```

Thanks for your attention!

About me:

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- How to reach me: chenxy346@gmail.com