The Relevance of t-Statistics for Small Sample Sizes

An Introductory Class to Higher Statistics

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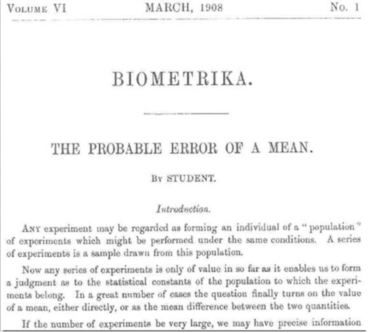
Outline

Content:

1. How was t-statistic developed?
2. Normal distribution vs. student t-distribution?
3. What is t-statistic?
4. How to calcuate a t-test?
5. What are the hypotheses and the assumptions?
6. How to interpret results?

A Fascinating Blend of Statistics & Industrial History

"The Guinness Brewery faced the problem of ensuring consistent quality in their beer. To achieve this, they needed to analyze small sample sizes of ingredients and processes, as it was impractical and wasteful to test entire batches. This required innovative statistical techniques to infer population parameters (e.g., the mean) from small samples."



VOLUME VI MARCH, 1908 No. 1

BIOMETRIKA.

THE PROBABLE ERROR OF A MEAN.

BY STUDENT.

Introduction.

ANY experiment may be regarded as forming an individual of a "population" of experiments which might be performed under the same conditions. A series of experiments is a sample drawn from this population.

Now any series of experiments is only of value in so far as it enables us to form a judgment as to the statistical constants of the population to which the experiments belong. In a great number of cases the question finally turns on the value of a mean, either directly, or as the mean difference between the two quantities.

If the number of experiments be very large, we may have precise information

Gossset'1908 (known as "Student" due to industrial secrets)

(Source: link)

Development

The problems of **small sample sizes**

1876

t-distribution as a posterior distribution

1895

t-distribution as Pearson type IV distribution

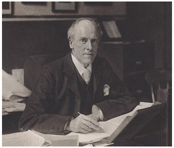
1908

Student t-distribution

"The Probable Error of a Mean" (Biometrika)

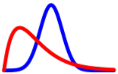
F. R. Helmert

J. Lüroth



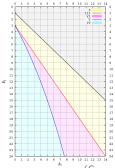
K. Pearson

W. S. Gosset



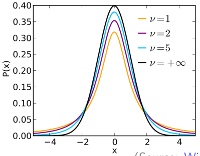
(Source: Wiki)

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$



(Source: Wiki)

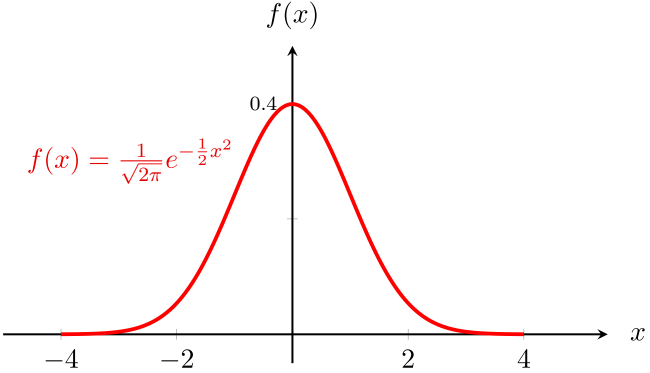
Diagram of the Pearson system



(Source: Wiki)

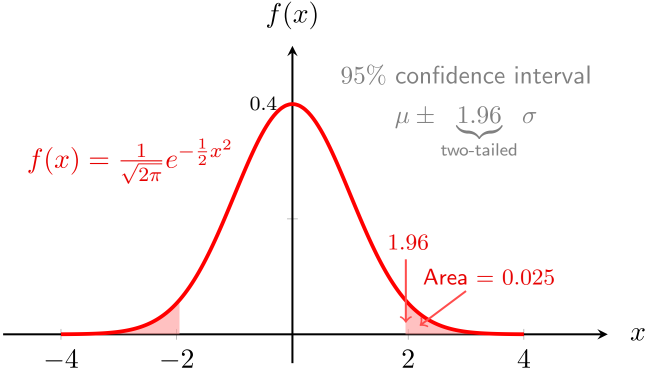
Probability density function

Revisiting Normal Distribution



Probability density function of the standard normal distribution

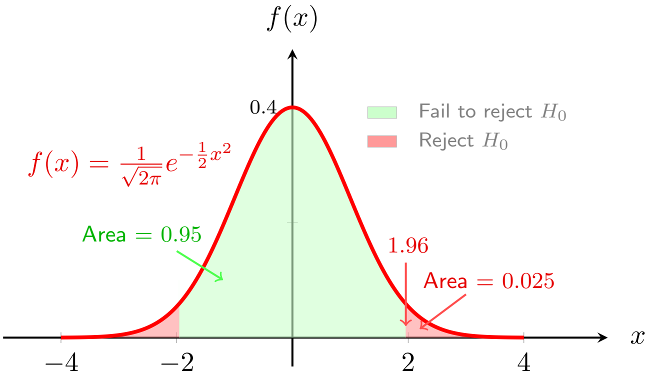
Revisiting Normal Distribution



Probability density function of the standard normal distribution

Connecting with Hypothesis Test

- Hypothesis test
  - Population: mean μ, standard deviation σ
  - Sample: mean $\bar{x}$, sample size n
  - Null hypothesis (H_0): The population mean is μ
  - z-statistic: $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ ($z \uparrow$ implies statistically significant difference)
- 95% confidence interval



Implementing z-Test

Problem Statement

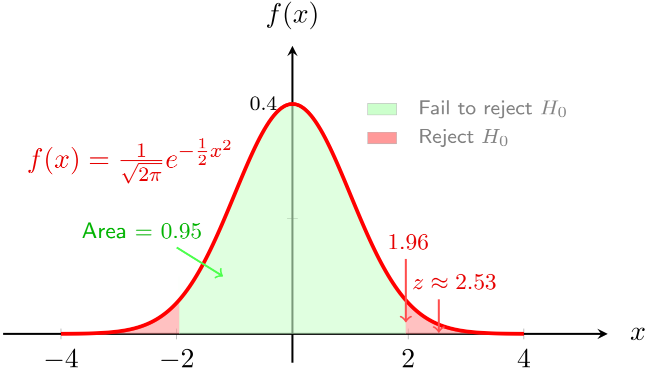
A company claims that the average daily energy consumption of households is 30 kWh with a population standard deviation of 5 kWh. A random sample of 40 households has an average daily energy consumption of 32 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

- $\bar{x} = 32$ (sample mean)
- $\mu = 30$ (population mean)
- $n = 40$ (sample size)
- $\sigma = 5$ (population standard deviation)

Steps:

1. Formulate Hypotheses
   - Null Hypothesis (H_0): The population mean is $\mu = 30\,\text{kWh}$.
   - Alternative Hypothesis (H_a): The population mean is $\mu \neq 30\,\text{kWh}$.
2. Use the z-test formula since the population standard deviation is known:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{32 - 30}{5/\sqrt{40}} = \frac{2}{5/6.32} = \frac{2}{0.79} \approx 2.53$$



Test statistic $|z| > 1.96 \Rightarrow$ Reject the null hypothesis

Student t-Distribution

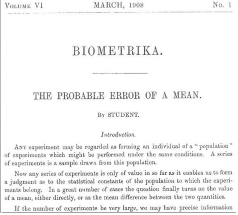
In the case of small sample sizes?

- Switch to student t-distribution and t-test

- Probability density function:

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

  - $x \in \mathbb{R}$: random variable
  - $\nu \in \mathbb{Z}^+$: degrees of freedom
  - $\Gamma(\cdot)$: Gamma function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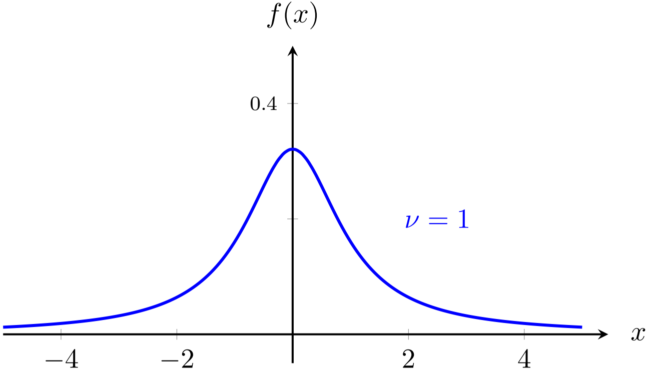
Gossset'1908 (known as "student")

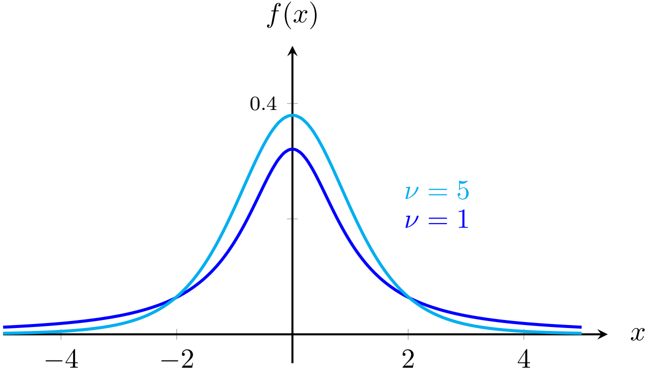
(Source: link)

Student t-Distribution



Student t-distribution of ν degrees of freedom

Student t-Distribution



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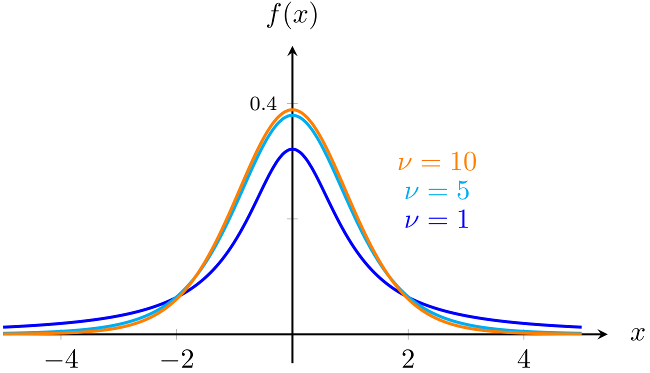

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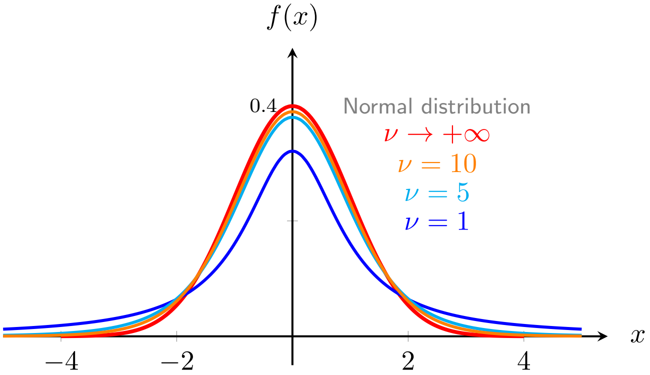

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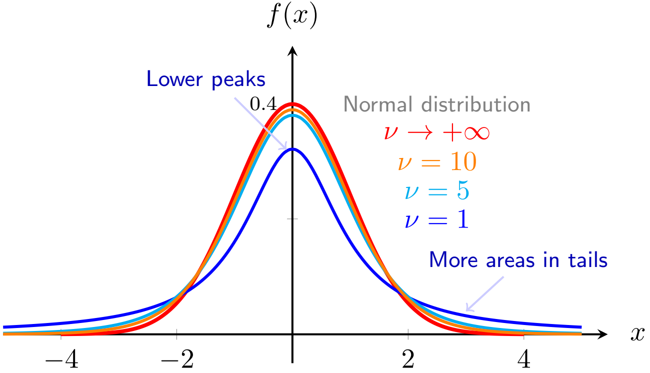


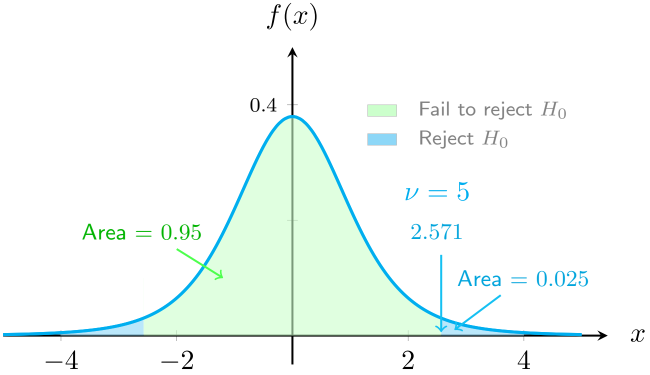
Student t-distribution of ν degrees of freedom

Definition of t-Statistic

- Formula of t-statistic for small sample sizes

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

  - Population: mean μ
  - Sample: mean $\bar{x}$, standard deviation s, sample size n (small value)
- A high absolute value of t suggests a statistically significant difference.



Implementing t-Test for Small Sample Size

Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh. A random sample of 6 households has an average daily energy consumption of 32 kWh, with a sample standard deviation of 4 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

- $\bar{x} = 32$ (sample mean)
- $s = 4$ (sample standard deviation)
- $n = 6$ (sample size)
- $\mu = 30$ (population mean)

Steps:

1. Formulate Hypotheses
   - Null Hypothesis (H_0): The population mean is $\mu = 30\,$kWh.
   - Alternative Hypothesis (H_a): The population mean is $\mu \neq 30\,$kWh.
2. Use the t-test formula since the population standard deviation is not known:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{32 - 30}{4/\sqrt{6}} = \frac{2}{4/2.449} \approx 1.22$$

Critical Values in t-Table

Small sample sizes

- Degrees of freedom for a t-test:

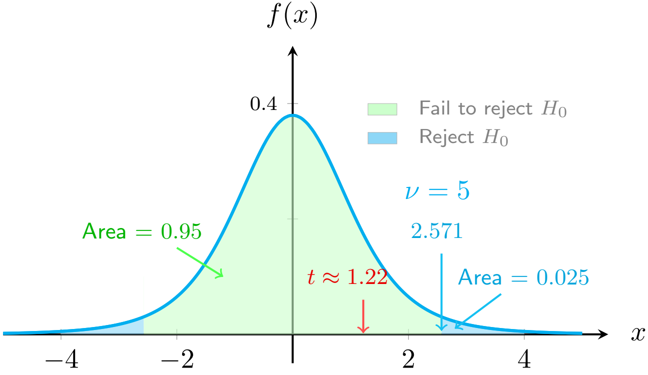
$$\nu = \underbrace{n}_{\text{sample size}} - 1 = 6 - 1 = 5$$

- t-distributions with ν degrees of freedom at a 95% confidence interval (two-tailed)

$\nu = 1$	$\nu = 5$	$\nu = 10$	$\nu \to +\infty$
12.706	2.571	2.228	1.960

- The critical t-value

$$t_{\nu,(1-0.95)/2} = t_{5,0.025} = 2.571$$



Test statistic $|t| < 2.571 \Rightarrow$ Fail to reject the null hypothesis

Implementing t-Test for Small Sample Size

Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh. A random sample of 6 households has an average daily energy consumption of 32 kWh, with a sample standard deviation of 6 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

Steps:

❷ Use the t-test formula:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{32 - 30}{6/\sqrt{6}} = \frac{2}{6/2.449} \approx 1.22$$

❸ Decision rule at a 95% confidence interval

- Reject H_0 if $|t| > 2.571$.
- Otherwise, fail to reject H_0.

❹ Interpretation

- The test statistic $|t| = 1.22 < 2.571$.
- Thus, we fail to reject the null hypothesis.
- There is not enough evidence to conclude that the average daily energy consumption differs from the company's claim of 30 kWh.

Normal Distribution vs. Student t-Distribution?

For the population mean μ (✓) and standard deviation σ (✓/✗)

- If population standard deviation σ is known

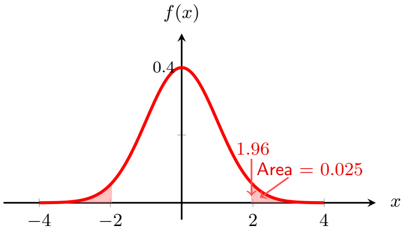
$$\bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

- Use z-test

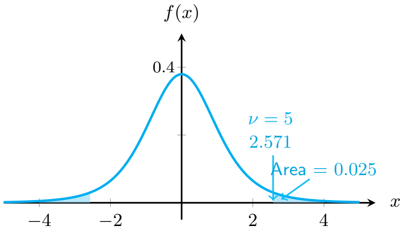
- If σ is unknown, using sample standard deviation s instead

$$\bar{x} \pm \, ? \times \frac{s}{\sqrt{n}}$$

- Use t-test



Standard normal distribution

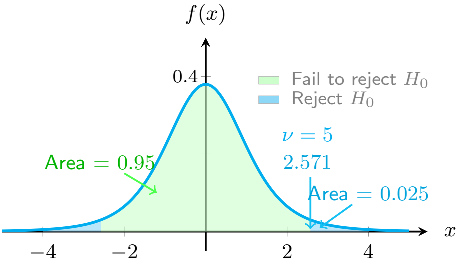


Student t-distribution

- **Heavy tail** in student t-distribution ($\nu = n - 1$ degrees of freedom) is important for small sample size n

Summary

- Student t-distribution of ν degrees of freedom



Student t-distribution

W. S. Gosset in Guinness

(Source: link)

- Population: mean μ (✓), standard deviation σ (✗)
- Sample: mean $\bar{x}$, standard deviation s, and small sample size n
- What is hypothesis test? 95% confidence interval: $\bar{x} \pm \underbrace{t_{\nu,0.025}}_{\nu=n-1} \times \frac{s}{\sqrt{n}}$
- What is t-statistic? How to calculate t-test? $t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$

Thank you!

Any Questions?

Slides: https://xinychen.github.io/slides/t_stat.pdf

About me:

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