Deblurring Images

Matrices, Spectra, and Filtering

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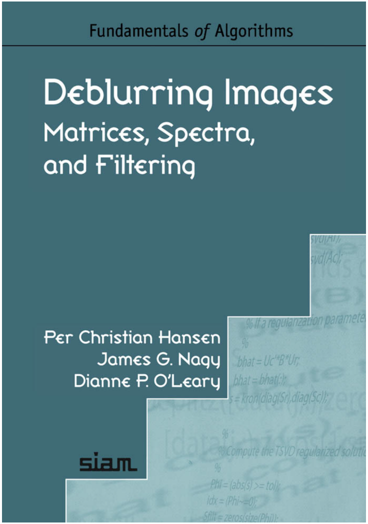


- **Chapter 1: The Image Deblurring Problem**
- Chapter 4: Structured Matrix Computations

The Image Deblurring Problem

About the image deblurring[1]:

- **[Significance]** Image deblurring is fundamental in making pictures sharp and useful.
- **[General idea]** Recovering the original and sharp image by using a mathematical model of blurring process.
- **[Fact]** No hope to recover the original image exactly!
- **[Technical goal]** Develop efficient and reliable algorithms for recovering as much information as possible from the given data.
- **[Representation]** A digital image is a two- or three-dimensional array of numbers representing intensities on a grayscale or color scale.

[1] The images and Matlab functions discussed in the book are available at https://archive.siam.org/books/fa03/.

The Image Deblurring Problem

A blurred picture and simple linear model.

- **Sharp image** vs. **blurred image**

- Notation: $\boldsymbol{X} \in \mathbb{R}^{m\times n}$ (desired **sharp** image) vs. $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ (recorded **blurred** image)
- A simple linear model:
  - Suppose the blurring of the columns in the image is independent of the blurring of the rows.
  - **Bilinear relationship**: $\boxed{\boldsymbol{A}_c\boldsymbol{X}\boldsymbol{A}_r^\top = \boldsymbol{B}}$

The Image Deblurring Problem

A first attempt at deblurring.

- Recall that the simple linear model:

$$\boldsymbol{A}_c \boldsymbol{X} \boldsymbol{A}_r^\top = \boldsymbol{B} \quad \Longrightarrow \quad \boldsymbol{X}_{\mathsf{naive}} = \boldsymbol{A}_c^{-1} \boldsymbol{B} (\boldsymbol{A}_r^\top)^{-1} \tag{1}$$

  ignores several types of errors.

- Let

$$\boldsymbol{B}_{\mathsf{exact}} = \boldsymbol{A}_c \boldsymbol{X} \boldsymbol{A}_r^\top \tag{2}$$

  be the ideal (noise-free) blurred image, ignoring all kinds of errors.

- Consider small random errors (noise) in the recorded blurred image:

$$\boldsymbol{B} = \boldsymbol{B}_{\mathsf{exact}} + \boldsymbol{E} = \boldsymbol{A}_c \boldsymbol{X} \boldsymbol{A}_r^\top + \boldsymbol{E} \tag{3}$$

  where $\boldsymbol{E} \in \mathbb{R}^{m \times n}$ is the **noise image**.

The Image Deblurring Problem

A first attempt at deblurring.

The naive reconstruction

Recall that

$$\begin{cases} \boldsymbol{X}_{\text{naive}} = \boldsymbol{A}_c^{-1}\boldsymbol{B}(\boldsymbol{A}_r^{\top})^{-1} \\ \boldsymbol{B} = \boldsymbol{B}_{\text{exact}} + \boldsymbol{E} = \boldsymbol{A}_c\boldsymbol{X}\boldsymbol{A}_r^{\top} + \boldsymbol{E} \end{cases} \tag{4}$$

we therefore have the naive reconstruction:

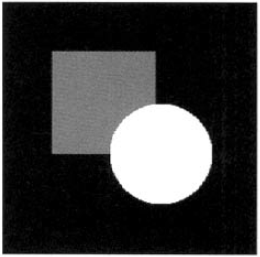
$$\begin{aligned} \boldsymbol{X}_{\text{naive}} &= \boldsymbol{A}_c^{-1}\boldsymbol{B}(\boldsymbol{A}_r^{\top})^{-1} \\ &= \boldsymbol{A}_c^{-1}\boldsymbol{B}_{\text{exact}}(\boldsymbol{A}_r^{\top})^{-1} + \boldsymbol{A}_c^{-1}\boldsymbol{E}(\boldsymbol{A}_r^{\top})^{-1} \\ &= \boldsymbol{X} + \boldsymbol{A}_c^{-1}\boldsymbol{E}(\boldsymbol{A}_r^{\top})^{-1} \end{aligned} \tag{5}$$

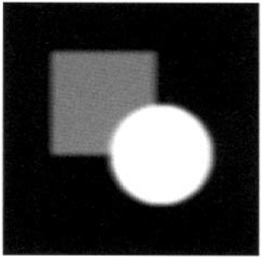
- The blurred image consists of two components: the first component is the **exact image**, and the second component is the **inverted noise**.

The Image Deblurring Problem

A first attempt at deblurring.

- A simple test: **Exact image** $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ vs. **blurred image** $\boldsymbol{B} \in \mathbb{R}^{m \times n}$





The Image Deblurring Problem

Lemma

For the simple model $\boldsymbol{B} = \boldsymbol{A}_c \boldsymbol{X} \boldsymbol{A}_r^\top + \boldsymbol{E}$, the relative error in the naive reconstruction $\boldsymbol{X}_{\text{naive}} = \boldsymbol{A}_c^{-1} \boldsymbol{B} (\boldsymbol{A}_r^\top)^{-1}$ satisfies

$$\frac{\|\boldsymbol{X}_{\text{naive}} - \boldsymbol{X}\|_F}{\|\boldsymbol{X}\|_F} \leq \text{cond}(\boldsymbol{A}_c) \cdot \text{cond}(\boldsymbol{A}_r) \cdot \frac{\|\boldsymbol{E}\|_F}{\|\boldsymbol{B}\|_F} \tag{6}$$

where $\|\cdot\|_F$ denotes the Frobenius norm[a], and $\text{cond}(\cdot)$ denotes the conditional number[b].

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[a] For any $\boldsymbol{X} \in \mathbb{R}^{m \times n}$, we have $\|\boldsymbol{X}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n x_{ij}^2}$.

[b] For any $\boldsymbol{A} \in \mathbb{R}^{N \times N}$ whose singular values are strictly positive, namely, $\sigma_1 \geq \cdots \geq \sigma_N > 0$, we have $\text{cond}(\boldsymbol{A}) = \sigma_1 / \sigma_N$.

The Image Deblurring Problem

Deblurring using a general linear model.

- In most situations, the blur is indeed **linear**, or at least well approximated by a linear model.
- A general linear model via **vectorization**.
    - Given sharp image $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ and blurred image $\boldsymbol{B} \in \mathbb{R}^{m \times n}$, since the blurring is assumed to be a linear operation, there must exist a large **blurring matrix** $\boldsymbol{A} \in \mathbb{R}^{N \times N}$ $(N = mn)$ such that

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{7}$$

with

$$\boldsymbol{x} = \mathrm{vec}(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_n \end{bmatrix} \in \mathbb{R}^N, \quad \boldsymbol{b} = \mathrm{vec}(\boldsymbol{B}) = \begin{bmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix} \in \mathbb{R}^N \tag{8}$$

    - The naive approach to image deblurring is simply to solve this linear algebraic system.

The Image Deblurring Problem

Deblurring using a general linear model.

The naive reconstruction (matrix-form)

Recall that

$$\begin{cases} \boldsymbol{X}_{\text{naive}} = \boldsymbol{A}_c^{-1}\boldsymbol{B}(\boldsymbol{A}_r^{\top})^{-1} \\ \boldsymbol{B} = \boldsymbol{B}_{\text{exact}} + \boldsymbol{E} = \boldsymbol{A}_c\boldsymbol{X}\boldsymbol{A}_r^{\top} + \boldsymbol{E} \end{cases} \tag{9}$$

we therefore have the naive reconstruction:

$$\begin{aligned} \boldsymbol{X}_{\text{naive}} &= \boldsymbol{A}_c^{-1}\boldsymbol{B}(\boldsymbol{A}_r^{\top})^{-1} \\ &= \boldsymbol{A}_c^{-1}\boldsymbol{B}_{\text{exact}}(\boldsymbol{A}_r^{\top})^{-1} + \boldsymbol{A}_c^{-1}\boldsymbol{E}(\boldsymbol{A}_r^{\top})^{-1} \\ &= \boldsymbol{X} + \boldsymbol{A}_c^{-1}\boldsymbol{E}(\boldsymbol{A}_r^{\top})^{-1} \end{aligned} \tag{10}$$

The naive reconstruction (vector-form)

Vectorize blurred image $\boldsymbol{B}$ and noise image $\boldsymbol{E}$ as $\boldsymbol{b}_{\text{exact}} = \text{vec}(\boldsymbol{B}_{\text{exact}}) = \boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{e} = \text{vec}(\boldsymbol{E})$, respectively, then we have

$$\boldsymbol{x}_{\text{naive}} = \boldsymbol{A}^{-1}\boldsymbol{b} = \boldsymbol{A}^{-1}\boldsymbol{b}_{\text{exact}} + \boldsymbol{A}^{-1}\boldsymbol{e} = \boldsymbol{x} + \boldsymbol{A}^{-1}\boldsymbol{e} \tag{11}$$

The Image Deblurring Problem

Deblurring using a general linear model.

- Relationship between matrix- and vector-form reconstruction:

$$\begin{aligned} \boldsymbol{X}_{\mathsf{naive}} &= \boldsymbol{A}_c^{-1}\boldsymbol{B}(\boldsymbol{A}_r^\top)^{-1} \\ \Longrightarrow \ \boldsymbol{x}_{\mathsf{naive}} &= (\boldsymbol{A}_r^{-1}\otimes\boldsymbol{A}_c^{-1})\boldsymbol{b} \\ &= (\boldsymbol{A}_r\otimes\boldsymbol{A}_c)^{-1}\boldsymbol{b} \end{aligned} \tag{12}$$

  it therefore demonstrates that $\boldsymbol{A}\triangleq\boldsymbol{A}_r\otimes\boldsymbol{A}_c$.

- Property of Kronecker product $\otimes$:

Proposition

Let $\boldsymbol{A}\in\mathbb{R}^{m\times m}$, $\boldsymbol{X}\in\mathbb{R}^{m\times n}$, and $\boldsymbol{B}\in\mathbb{R}^{n\times n}$ be three matrices commensurate from multiplication in that order, then it holds that

$$\mathsf{vec}(\boldsymbol{AXB})=(\boldsymbol{B}^\top\otimes\boldsymbol{A})\mathsf{vec}(\boldsymbol{X}) \tag{13}$$

The Image Deblurring Problem

Deblurring using a general linear model.

Singular value decomposition (SVD)

For any $\boldsymbol{A} \in \mathbb{R}^{N\times N}$ whose singular values are strictly positive, we have

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top = \sum_{i=1}^{N} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^\top \quad \Longrightarrow \quad \boldsymbol{A}^{-1} = \sum_{i=1}^{N} \frac{1}{\sigma_i} \boldsymbol{u}_i \boldsymbol{v}_i^\top \tag{14}$$

The naive reconstruction with SVD

The naive reconstruction can be written as follows,

$$\boldsymbol{x}_{\text{naive}} = \boldsymbol{A}^{-1}\boldsymbol{b} = \boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^\top \boldsymbol{b} = \sum_{i=1}^{N} \frac{\boldsymbol{u}_i^\top \boldsymbol{b}}{\sigma_i} \boldsymbol{v}_i \tag{15}$$

in which the inverted noise is

$$\boldsymbol{A}^{-1}\boldsymbol{e} = \boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^\top \boldsymbol{e} = \sum_{i=1}^{N} \frac{\boldsymbol{u}_i^\top \boldsymbol{e}}{\sigma_i} \boldsymbol{v}_i \tag{16}$$

The Image Deblurring Problem

Deblurring using a general linear model.

- Recall that the inverted noise is

$$\boldsymbol{A}^{-1}\boldsymbol{e} = \boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^\top\boldsymbol{e} = \sum_{i=1}^{N} \frac{\boldsymbol{u}_i^\top \boldsymbol{e}}{\sigma_i}\boldsymbol{v}_i$$

- Properties for image deblurring problems:
    - The error components $|\boldsymbol{u}_i^\top \boldsymbol{e}|$ are small and typically of roughly the same order of magnitude for all i.
    - The singular values decay to a value very close to zero. As a consequence, the condition number $\text{cond}(\boldsymbol{A}) = \sigma_1/\sigma_N$ is very large, indicating that **the solution is very sensitive to perturbation and rounding errors**.
    - **The singular vectors corresponding to the smaller singular values typically represent high-frequency information**. That is, as i increases, the vectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ tend to have more sign changes.

The Image Deblurring Problem

Deblurring using a general linear model.

- Recall that the inverted noise is

$$\boldsymbol{A}^{-1}\boldsymbol{e} = \boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^\top \boldsymbol{e} = \sum_{i=1}^{N} \frac{\boldsymbol{u}_i^\top \boldsymbol{e}}{\sigma_i} \boldsymbol{v}_i$$

Remark

For $\boldsymbol{A}^{-1}\boldsymbol{e}$, the quantities $\boldsymbol{u}_i^\top \boldsymbol{e}/\sigma_i$ are the expansion coefficients for the basis vectors $\boldsymbol{v}_i$. When these quantities are small in magnitude, the solution has very little contribution from $\boldsymbol{v}_i$, but when we divide by a small singular values such as σ_N, we greatly magnify the corresponding error component $\boldsymbol{u}_N^\top \boldsymbol{e}$ which in turn contributes a large multiple of the high-frequency information contained in $\boldsymbol{v}_N$ to the reconstruction solution.

- Thus, we can remove the high-frequency components that are dominated by error.

The Image Deblurring Problem

Deblurring using a general linear model.

- The naive reconstruction with SVD:

$$\boldsymbol{x}_{\mathsf{naive}} = \sum_{i=1}^{N} \frac{\boldsymbol{u}_i^\top \boldsymbol{b}}{\sigma_i} \boldsymbol{v}_i \tag{17}$$

- The truncated expansion with $k < N, k \in \mathbb{N}^+$:

$$\boldsymbol{x}_k = \sum_{i=1}^{k} \frac{\boldsymbol{u}_i^\top \boldsymbol{b}}{\sigma_i} \boldsymbol{v}_i \tag{18}$$

  which is indeed a reduced-rank linear model.

- We may wonder if a different value for k will produce a better reconstruction!

Structured Matrix Computations

- A general linear model:

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{e} \tag{19}$$

with

$$\begin{cases} \boldsymbol{b} = \mathsf{vec}(\boldsymbol{B}) \in \mathbb{R}^N & \text{(blurred image)} \\ \boldsymbol{x} = \mathsf{vec}(\boldsymbol{X}) \in \mathbb{R}^N & \text{(sharp image)} \\ \boldsymbol{e} = \mathsf{vec}(\boldsymbol{E}) \in \mathbb{R}^N & \text{(noise image)} \\ \boldsymbol{A} \in \mathbb{R}^{N\times N} & \text{(blurring matrix)} \end{cases}$$

- The deblurring algorithms use certain orthogonal or unitary decompositions of $\boldsymbol{A}$.
  - SVD: $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$ vs. spectral decomposition[2]: $\boldsymbol{A} = \tilde{\boldsymbol{U}}\boldsymbol{\Lambda}\tilde{\boldsymbol{U}}^H$
  - If $\boldsymbol{A}$ has real entries, then the elements in the matrices of the SVD will be real, but the entries in the spectral decomposition may be complex.

[2] A matrix is unitary if $\tilde{\boldsymbol{U}}^H\tilde{\boldsymbol{U}} = \tilde{\boldsymbol{U}}\tilde{\boldsymbol{U}}^H = \boldsymbol{I}$ where $\tilde{\boldsymbol{U}}^H = \mathsf{conj}(\tilde{\boldsymbol{U}})^\top$ is the complex conjugate transpose of $\tilde{\boldsymbol{U}}$. $\boldsymbol{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\boldsymbol{A}$.

Structured Matrix Computations

Basic structures.

- Convolution is a mathematical operation.
- If $p(s)$ and $x(s)$ are **continuous** functions, then the convolution of $p(s)$ and $x(s)$ is a function $b(s)$ having the form

$$b(s) = \int_{-\infty}^{\infty} p(s-t)x(t)dt \tag{20}$$

  each values of $b(s)$ is essentially a weighted average of the values of $x(s)$, where the weights are given by $p(s)$.
- The **discrete** version of convolution is a summation over a finite number of terms.

Structured Matrix Computations

Basic structures[3].

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[3] Lecture 14: Structured matrices, FFT, convolutions, Toeplitz matrices