Laplacian Convolutional Representation for Traffic Time Series Imputation

Xinyu Chen

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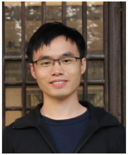

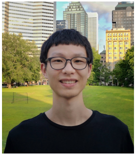

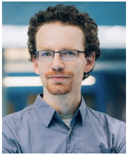

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Preprint:

- X. Chen, Z. Cheng, N. Saunier, L. Sun (2022). Laplacian convolutional representation for traffic time series imputation. arXiv preprint arXiv:2212.01529. https://xinychen.github.io/papers/Laplacian_convolution.pdf

GitHub repository:

- **transdim**: Machine learning for spatiotemporal traffic data imputation and forecasting. (1,000+ stars & 270+ forks on GitHub) https://github.com/xinychen/transdim

Slides:

- https://xinychen.github.io/slides/LCR.pdf

Outline

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  - Time Series Imputation
  - Speed Field Reconstruction
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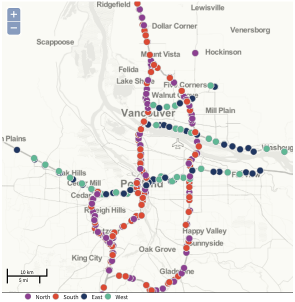
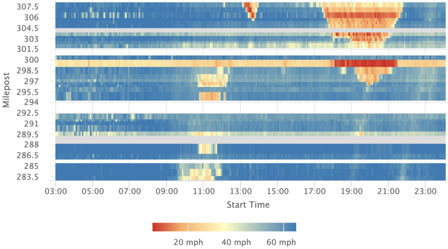
  - Model Description
  - Solution Algorithm
- **Experiments**
  - Univariate Traffic Time Series Imputation
  - Speed Field Reconstruction
- **Conclusion**

Motivation

- Portland highway traffic flow data[1]



Highway network & sensor locations

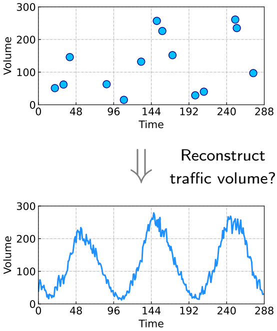


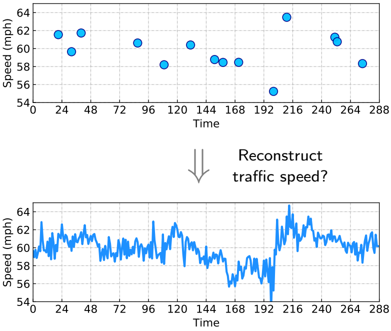
Traffic speed field

- Speed field $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ (N locations & T time steps)
- Speed field shows strong spatial/temporal dependencies

[1] https://portal.its.pdx.edu/home

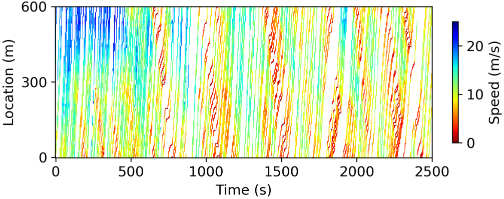
Motivation



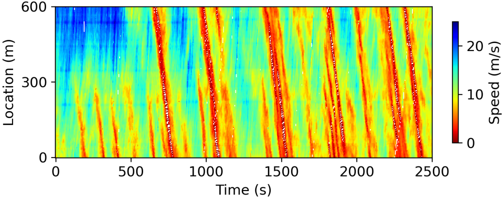


- How to utilize the global trends of traffic time series?
- How to produce local consistency of traffic data?

Motivation



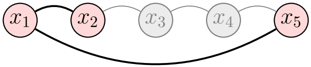
200-by-500 matrix (NGSIM) $\Downarrow$ Reconstruct speed field from 20% sparse trajectories?



- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal local dependencies?

Revisit Laplacian Matrix & Circular Convolution

- Intuition of (circulant) Laplacian matrix.



Undirected and circulant graph

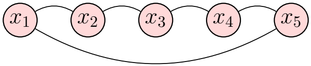
Modeling ⟶

$$\boldsymbol{L} = \begin{bmatrix} \mathbf{2} & -1 & 0 & 0 & -1 \\ \mathbf{-1} & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ \mathbf{-1} & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

Revisit Laplacian Matrix & Circular Convolution

- Intuition of (circulant) Laplacian matrix.



Undirected and circulant graph

Modeling $\longrightarrow$

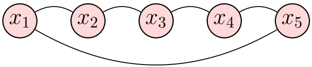
$$\boldsymbol{L} = \begin{bmatrix} \mathbf{2} & -1 & 0 & 0 & -1 \\ \mathbf{-1} & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ \mathbf{-1} & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

Revisit Laplacian Matrix & Circular Convolution

Reformulate Laplacian regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.



Undirected and circulant graph

Modeling $\longrightarrow$

$$\boldsymbol{L}=\begin{bmatrix}2 & -1 & 0 & 0 & -1\\ -1 & 2 & -1 & 0 & 0\\ 0 & -1 & 2 & -1 & 0\\ 0 & 0 & -1 & 2 & -1\\ -1 & 0 & 0 & -1 & 2\end{bmatrix}$$

(Circulant) Laplacian matrix

- Laplacian kernel: $\boldsymbol{\ell}=(2,-1,0,0,-1)^\top$.

$$\boldsymbol{L}\boldsymbol{x}=\begin{bmatrix}2 & -1 & 0 & 0 & -1\\ -1 & 2 & -1 & 0 & 0\\ 0 & -1 & 2 & -1 & 0\\ 0 & 0 & -1 & 2 & -1\\ -1 & 0 & 0 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\begin{bmatrix}2\\ -1\\ 0\\ 0\\ -1\end{bmatrix}\star\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\boldsymbol{\ell}\star\boldsymbol{x}$$

where $\star$ denotes the ciruclar convolution.

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}(\boldsymbol{x})=\frac{1}{2}\|\boldsymbol{L}\boldsymbol{x}\|_2^2=\frac{1}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2$$

Revisit Laplacian Matrix & Circular Convolution

Reformulate Laplacian regularization with circular convolution.

- Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \cdots, -1}_{\tau}, 0, \cdots, 0, \underbrace{-1, \cdots, -1}_{\tau})^\top \in \mathbb{R}^T$$

  for any time series $\boldsymbol{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$.

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}_\tau(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{L}\boldsymbol{x}\|_2^2 = \frac{1}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2$$

- Property with discrete Fourier transform (denoted by $\mathcal{F}(\cdot)$)[2]:

$$\mathcal{R}_\tau(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 = \frac{1}{2T}\|\mathcal{F}(\boldsymbol{\ell}) \circ \mathcal{F}(\boldsymbol{x})\|_2^2$$

[2] It refers to the Convolution theorem.

Circulant Matrix Nuclear Norm Minimization

Circulant Matrix Nuclear Norm Minimization (CircNNM)

For any partially observed time series $\boldsymbol{y} \in \mathbb{R}^T$ with observed index set Ω, the optimization problem of CircNNM for reconstructing time series is given by

$$\begin{aligned} \min_{\boldsymbol{x}} \; & \|\mathcal{C}(\boldsymbol{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_{\Omega}(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

where $\mathcal{C} : \mathbb{R}^T \to \mathbb{R}^{T \times T}$ denotes the circulant operator. $\|\cdot\|_*$ denotes the nuclear norm of matrix, namely, the sum of singular values.

- An important property:
  $$\|\mathcal{C}(\boldsymbol{x})\|_* = \|\mathcal{F}(\boldsymbol{x})\|_1$$
- CircNNM shows an efficient FFT[3] implementation in $\mathcal{O}(T \log T)$ time (Liu'22, Liu & Zhang'23).

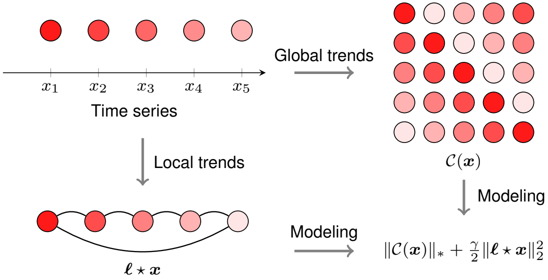
[3] Fast Fourier Transform (FFT).

Laplacian Convolutional Representation

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\boldsymbol{y} \in \mathbb{R}^T$ with observed index set Ω, LCR utilizes circulant matrix and Laplacian kernel to characterize **global and local trends** in time series, respectively, i.e.,

$$\min_{\boldsymbol{x}} \; \|\mathcal{C}(\boldsymbol{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\boldsymbol{x})$$
$$\text{s.t. } \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon$$



Laplacian Convolutional Representation

- LCR model:

$$\begin{aligned}\min_{\boldsymbol{x}}\ & \|\mathcal{C}(\boldsymbol{x})\|_* + \gamma\cdot\mathcal{R}_\tau(\boldsymbol{x})\\ \text{s.t. } & \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2\leq\epsilon\end{aligned}$$

- Augmented Lagrangian function:

$$\mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) = \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2 + \langle\boldsymbol{w},\boldsymbol{x}-\boldsymbol{z}\rangle + \frac{\eta}{2}\|\mathcal{P}_\Omega(\boldsymbol{z}-\boldsymbol{y})\|_2^2$$

where $\boldsymbol{w}\in\mathbb{R}^T$ is the Lagrange multiplier, and $\langle\cdot,\cdot\rangle$ denotes the inner product.

Laplacian Convolutional Representation

- LCR model:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\mathcal{C}(\boldsymbol{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\boldsymbol{x}) \\ \text{s.t.} \quad & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

- Augmented Lagrangian function:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) = \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x} - \boldsymbol{z}\|_2^2 + \langle \boldsymbol{w}, \boldsymbol{x} - \boldsymbol{z} \rangle + \frac{\eta}{2}\|\mathcal{P}_\Omega(\boldsymbol{z} - \boldsymbol{y})\|_2^2$$

where $\boldsymbol{w} \in \mathbb{R}^T$ is the Lagrange multiplier, and $\langle \cdot, \cdot \rangle$ denotes the inner product.

- The ADMM scheme:

$$\begin{cases} \boldsymbol{x} := \arg\min_{\boldsymbol{x}} \ \mathcal{L}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) \\ \boldsymbol{z} := \arg\min_{\boldsymbol{z}} \ \mathcal{L}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) \\ \quad = \frac{1}{\lambda + \eta}\mathcal{P}_\Omega(\lambda \boldsymbol{x} + \boldsymbol{w} + \eta \boldsymbol{y}) + \frac{1}{\lambda}\mathcal{P}_\Omega^\perp(\lambda \boldsymbol{x} + \boldsymbol{w}) \\ \boldsymbol{w} := \boldsymbol{w} + \lambda(\boldsymbol{x} - \boldsymbol{z}) \end{cases}$$

Laplacian Convolutional Representation

- Optimize $\boldsymbol{x}$ via FFT (in $\mathcal{O}(T\log T)$ time):

$$\begin{aligned} \boldsymbol{x} &:= \arg\min_{\boldsymbol{x}}\ \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}+\boldsymbol{w}/\lambda\|_2^2 \\ \Longrightarrow \hat{\boldsymbol{x}} &:= \arg\min_{\hat{\boldsymbol{x}}}\ \|\hat{\boldsymbol{x}}\|_1 + \frac{\gamma}{2T}\|\hat{\boldsymbol{\ell}}\circ\hat{\boldsymbol{x}}\|_2^2 + \frac{\lambda}{2T}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{z}}+\hat{\boldsymbol{w}}/\lambda\|_2^2 \end{aligned}$$

where we introduce $\{\hat{\boldsymbol{\ell}},\hat{\boldsymbol{x}},\hat{\boldsymbol{z}},\hat{\boldsymbol{w}}\}\triangleq\mathcal{F}\{\boldsymbol{\ell},\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}\}$ (i.e., FFT).

Laplacian Convolutional Representation

- Optimize $\boldsymbol{x}$ via FFT (in $\mathcal{O}(T\log T)$ time):

$$\boldsymbol{x} := \arg\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}+\boldsymbol{w}/\lambda\|_2^2$$

$$\Longrightarrow \hat{\boldsymbol{x}} := \arg\min_{\hat{\boldsymbol{x}}} \ \|\hat{\boldsymbol{x}}\|_1 + \frac{\gamma}{2T}\|\hat{\boldsymbol{\ell}}\circ\hat{\boldsymbol{x}}\|_2^2 + \frac{\lambda}{2T}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{z}}+\hat{\boldsymbol{w}}/\lambda\|_2^2$$

where we introduce $\{\hat{\boldsymbol{\ell}},\hat{\boldsymbol{x}},\hat{\boldsymbol{z}},\hat{\boldsymbol{w}}\}\triangleq\mathcal{F}\{\boldsymbol{\ell},\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}\}$ (i.e., FFT).

ℓ_1-norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of ℓ_1-norm minimization in complex space:

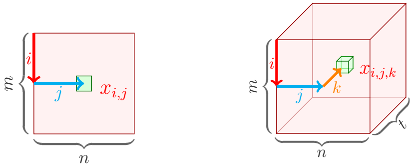
$$\min_{\hat{\boldsymbol{x}}} \ \|\hat{\boldsymbol{x}}\|_1 + \frac{\delta}{2}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{h}}\|_2^2$$

with complex-valued $\hat{\boldsymbol{x}},\hat{\boldsymbol{h}}\in\mathbb{C}^T$ and weight parameter δ, element-wise, the solution is given by

$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|}\cdot\max\{0,|\hat{h}_t|-1/\delta\}, t=1,\ldots,T.$$

Laplacian Convolutional Representation

- What is tensor? $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ vs. $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{m \times n \times t}$



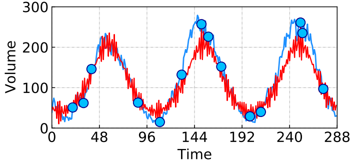
Two-Dimensional LCR (LCR-2D)

For any partially observed time series $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ with observed index set Ω, LCR can be formulated as follows,

$$\begin{aligned} \min_{\boldsymbol{X}} \; & \|\mathcal{C}(\boldsymbol{X})\|_* + \frac{\gamma}{2}\|(\boldsymbol{\ell}_s \boldsymbol{\ell}^\top) \star \boldsymbol{X}\|_F^2 \\ \text{s.t. } & \|\mathcal{P}_\Omega(\boldsymbol{X} - \boldsymbol{Y})\|_F \leq \epsilon \end{aligned}$$

where $\mathcal{C} : \mathbb{R}^{N \times T} \to \mathbb{R}^{N \times N \times T \times T}$ denotes the circulant operator.

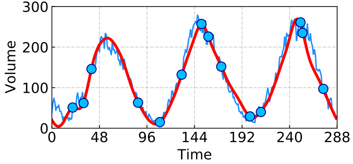
Experiments



CircNNM:

$$\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_*$$
$$\text{s. t.} \ \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon$$

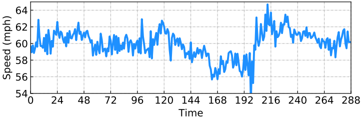
$\Downarrow$ Plus temporal regularization (TR)



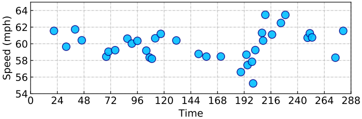
LCR:

$$\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2$$
$$\text{s. t.} \ \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon$$

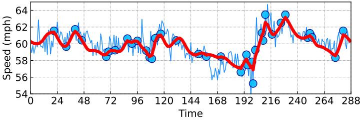
Experiments

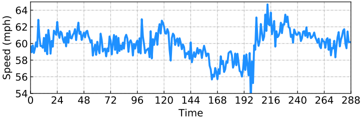


⇓ Mask **90%** observations

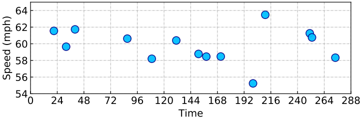


⇓ Reconstruct time series

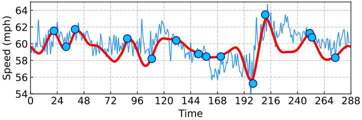




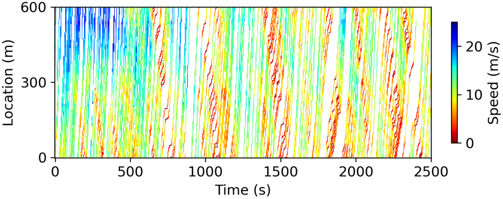
⇓ Mask **95%** observations



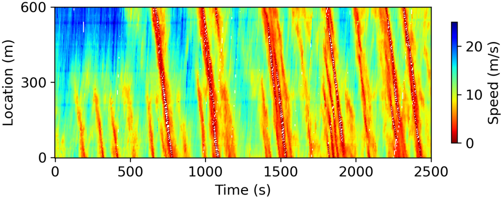
⇓ Reconstruct time series



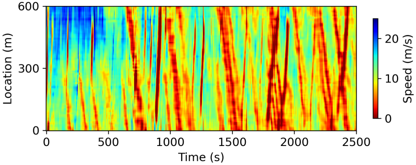
Experiments



200-by-500 matrix (NGSIM) $\Downarrow$ Reconstruct speed field from 20% sparse trajectories?



Experiments

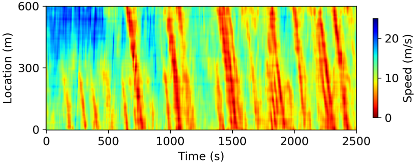


TR:

$$\min_{\boldsymbol{X}} \ \mathcal{R}_{\tau}(\boldsymbol{X})$$

$$\text{s. t. } \ \|\mathcal{P}_{\Omega}(\boldsymbol{X}-\boldsymbol{Y})\|_F \leq \epsilon$$

$\Downarrow$ Plus CircNNM

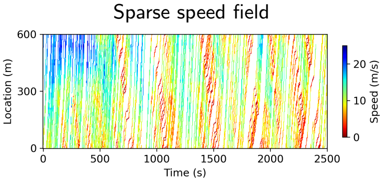


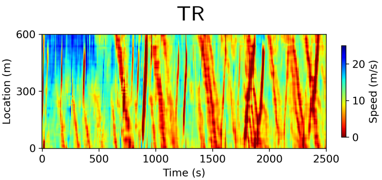
LCR:

$$\min_{\boldsymbol{X}} \ \|\mathcal{C}(\boldsymbol{X})\|_* + \gamma \cdot \mathcal{R}_{\tau}(\boldsymbol{X})$$

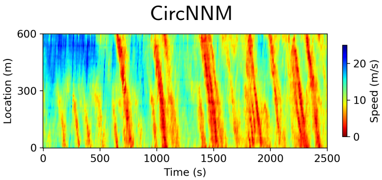
$$\text{s. t. } \ \|\mathcal{P}_{\Omega}(\boldsymbol{X}-\boldsymbol{Y})\|_F \leq \epsilon$$

Experiments

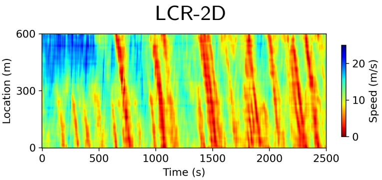




MAPE = 46.94% & RMSE = 4.34 m/s



MAPE = 43.51% & RMSE = 1.65 m/s



MAPE = **41.29%** & RMSE = **1.55 m/s**

Conclusion

- **(Starting point)** How to impute traffic time series?

  ✓ Low-rank models ✓ Temporal regularization

- **(Solution)** Time series trend modeling in the low-rank framework?
  - Global time series trend modeling (low-rank model):

$$\begin{aligned} \min_{\boldsymbol{x}} \; & \|\mathcal{C}(\boldsymbol{x})\|_* \\ \text{s.t.} \; & \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

  - Local time series trend modeling (temporal regularization):

$$\mathcal{R}_\tau(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2$$

- **(Highlight)** A unified framework with the FFT implementation.

References

A short list:

- **[Liu'22]** G. Liu (2022). Time series forecasting via learning convolutionally low-rank models. IEEE Transactions on Information Theory, 68(5): 3362–3380.
- **[Liu & Zhang'23]** G. Liu and W. Zhang (2023). Recovery of future data via convolution nuclear norm minimization. IEEE Transactions on Information Theory, 69(1): 650–665.

Thanks for your attention!

Any Questions?

About me:

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