Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 9: Introduction to Applied Linear Algebra: Part II

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Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

  "`Class Participation Quiz 19`"

  Time slot: **2:30PM – 3:00PM**

  on Canvas.

Adjoint of 3×3 Matrices

Adjoint matrix

- **Purpose**: It is primarily used to calculate the inverse of a matrix without using Gauss elimination.
- The relationship between a matrix, its adjoint, and its inverse is defined by:

$$\boldsymbol{A}^{-1} = \frac{\operatorname{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})}$$

- **Constraint**: This formula only works if $\det(\boldsymbol{A}) \neq 0$.

Adjoint of 3×3 Matrices

Adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$:

❶ **Calculate minors**: Determinant of the 2×2 matrix that remains after deleting the i-th row and j-th column, e.g.,

$$m_{11} = \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) = ei - fh$$

deleting row 1 and column 1.

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deleting row 1 and column 1.

$$m_{12} = \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) = di - fg$$

deleting row 1 and column 2.

$$m_{13} = \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right) = dh - eg$$

deleting row 1 and column 3.

Adjoint of 3×3 Matrices

Adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$:

❷ Apply signs to create the **cofactor matrix**: The cofactor c_{ij} is the minor multiplied by $(-1)^{i+j}$.

$$c_{ij} = (-1)^{i+j} \cdot m_{ij}$$

Adjoint of 3×3 Matrices

Adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$:

❷ Apply signs to create the **cofactor matrix**: The cofactor c_{ij} is the minor multiplied by $(-1)^{i+j}$.

$$c_{ij} = (-1)^{i+j} \cdot m_{ij}$$

- This creates a "checkerboard" pattern of **signs**:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

- The resulting **cofactor matrix** C is:

$$\boldsymbol{C} = \begin{bmatrix} +m_{11} & -m_{12} & +m_{13} \\ -m_{21} & +m_{22} & -m_{23} \\ +m_{31} & -m_{32} & +m_{33} \end{bmatrix}$$

Adjoint of 3×3 Matrices

Adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$:

❸ Transpose of cofactor matrix $\boldsymbol{C}$:

$$\operatorname{adj}(\boldsymbol{A}) = \boldsymbol{C}^\top = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix}$$

switching the rows and columns.

Adjoint of 3×3 Matrices

Rethink one question: How to compute

$$\begin{aligned} 8x_1 + 2x_2 - 2x_3 &= -2 \\ 10x_1 + 2x_2 + 4x_3 &= 4 \\ 12x_1 + 2x_2 + 2x_3 &= 6 \end{aligned}$$

Write down the adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} 8 & 2 & -2 \\ 10 & 2 & 4 \\ 12 & 2 & 2 \end{bmatrix}$:

Adjoint of 3×3 Matrices

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Write down the adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} 8 & 2 & -2 \\ 10 & 2 & 4 \\ 12 & 2 & 2 \end{bmatrix}$:

1. **Calculate minors**:

$$\begin{aligned} m_{11} &= 2 \times 2 - 4 \times 2 = -4, & m_{12} &= 10 \times 2 - 4 \times 12 = -28, \\ m_{13} &= 10 \times 2 - 2 \times 12 = -4, & m_{21} &= 2 \times 2 - (-2) \times 2 = 8, \\ m_{22} &= 8 \times 2 - (-2) \times 12 = 40, & m_{23} &= 8 \times 2 - 2 \times 12 = -8, \\ m_{31} &= 2 \times 4 - (-2) \times 2 = 12, & m_{32} &= 8 \times 4 - (-2) \times 10 = 52, \\ m_{33} &= 8 \times 2 - 2 \times 10 = -4 \end{aligned}$$

Adjoint of 3×3 Matrices

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Write down the adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} 8 & 2 & -2 \\ 10 & 2 & 4 \\ 12 & 2 & 2 \end{bmatrix}$:

❷ **Create cofactor matrix**:

$$\boldsymbol{C} = \begin{bmatrix} +m_{11} & -m_{12} & +m_{13} \\ -m_{21} & +m_{22} & -m_{23} \\ +m_{31} & -m_{32} & +m_{33} \end{bmatrix} = \begin{bmatrix} -4 & -(-28) & -4 \\ -(8) & 40 & -(-8) \\ 12 & -(52) & -4 \end{bmatrix}$$

Adjoint of 3×3 Matrices

Rethink one question: How to compute

$$
\begin{aligned}
8x_1 + 2x_2 - 2x_3 &= -2 \\
10x_1 + 2x_2 + 4x_3 &= 4 \\
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\end{aligned}
$$

Write down the adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} 8 & 2 & -2 \\ 10 & 2 & 4 \\ 12 & 2 & 2 \end{bmatrix}$:

❷ **Create cofactor matrix**:

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\boldsymbol{C} = \begin{bmatrix} +m_{11} & -m_{12} & +m_{13} \\ -m_{21} & +m_{22} & -m_{23} \\ +m_{31} & -m_{32} & +m_{33} \end{bmatrix} = \begin{bmatrix} -4 & -(-28) & -4 \\ -(8) & 40 & -(-8) \\ 12 & -(52) & -4 \end{bmatrix}
$$

❸ **Transpose** of cofactor matrix:

$$
\operatorname{adj}(\boldsymbol{A}) = \boldsymbol{C}^\top = \begin{bmatrix} -4 & -8 & 12 \\ 28 & 40 & -52 \\ -4 & 8 & -4 \end{bmatrix}
$$

Adjoint of 3×3 Matrices

Rethink one question: How to compute

$$\begin{aligned} 8x_1 + 2x_2 - 2x_3 &= -2 \\ 10x_1 + 2x_2 + 4x_3 &= 4 \\ 12x_1 + 2x_2 + 2x_3 &= 6 \end{aligned} \quad \Rightarrow \quad \boldsymbol{A} = \begin{bmatrix} 8 & 2 & -2 \\ 10 & 2 & 4 \\ 12 & 2 & 2 \end{bmatrix}$$

- Adjoint matrix:

$$\mathrm{adj}(\boldsymbol{A}) = \begin{bmatrix} -4 & -8 & 12 \\ 28 & 40 & -52 \\ -4 & 8 & -4 \end{bmatrix}$$

- **Determinant**:

$$\begin{aligned} \det(\boldsymbol{A}) &= 8 \times (2 \times 2 - 4 \times 2) - 2 \times (10 \times 2 - 4 \times 12) - 2 \times (10 \times 2 - 2 \times 12) \\ &= 8 \times (-4) - 2 \times (-28) - 2 \times (-4) = -32 + 56 + 8 = 32 \end{aligned}$$

Adjoint of 3×3 Matrices

Solve linear systems:

$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{A}^{-1}\boldsymbol{b} \\
&= \frac{\operatorname{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})}\boldsymbol{b} \\
&= \frac{1}{32}\begin{bmatrix} -4 & -8 & 12 \\ 28 & 40 & -52 \\ -4 & 8 & -4 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} \\
&= \frac{1}{32}\begin{bmatrix} (-4)\times(-2)+(-8)\times 4+12\times 6 \\ 28\times(-2)+40\times 4+(-52)\times 6 \\ (-4)\times(-2)+8\times 4+(-4)\times 6 \end{bmatrix} \\
&= \frac{1}{32}\begin{bmatrix} 48 \\ -208 \\ 16 \end{bmatrix} \\
&= \begin{bmatrix} 1.5 \\ -6.5 \\ 0.5 \end{bmatrix}
\end{aligned}
$$

Inverse of 3×3 Matrices

- Definition:

$$\boldsymbol{A}^{-1} = \frac{\operatorname{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})}$$

- Requirement:

The inverse $\boldsymbol{A}^{-1}$ exists if and only if $\det(\boldsymbol{A}) \neq 0$.

- Property:

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

Orthonormal Matrix

- What happened when

$$A^\top A = AA^\top = I$$

A square matrix A is orthogonal matrix if its transpose is equal to its inverse

$$A^\top = A^{-1}.$$

Orthonormal Vectors

Definition: A collection of vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ is orthonormal if

$$\boldsymbol{a}_i^\top \boldsymbol{a}_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

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❶ The ℓ_2-norm of vectors is 1:

$$\|\boldsymbol{a}_i\|_2 = 1, \quad i = 1, 2, \ldots, n$$

❷ Vectors are mutually orthogonal (i.e., inner product is zero):

$$\boldsymbol{a}_i^\top \boldsymbol{a}_j = 0, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n, \quad i \neq j$$

Orthonormal Vectors

Example:

- Three vectors:

$$\boldsymbol{a}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{a}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{a}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

- They are **orthonormal** because

  ❶ ℓ_2-norm: $\|\boldsymbol{a}_1\|_2 = \|\boldsymbol{a}_2\|_2 = \|\boldsymbol{a}_3\|_2 = 1$

  ❷ inner product: $\boldsymbol{a}_1^\top \boldsymbol{a}_2 = \boldsymbol{a}_1^\top \boldsymbol{a}_3 = \boldsymbol{a}_2^\top \boldsymbol{a}_3 = 0$

Matrix with Orthonormal Columns

If

$$A^\top A = \begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix}^\top \begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{a}_1^\top \boldsymbol{a}_1 & \boldsymbol{a}_1^\top \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_1^\top \boldsymbol{a}_n \\ \boldsymbol{a}_2^\top \boldsymbol{a}_1 & \boldsymbol{a}_2^\top \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_2^\top \boldsymbol{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_n^\top \boldsymbol{a}_1 & \boldsymbol{a}_n^\top \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n^\top \boldsymbol{a}_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \boldsymbol{I}$$

then, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ has orthonormal columns.

A square real matrix with orthonormal columns is **orthogonal matrix**.

Matrix-Vector Multiplication

If $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ has orthonormal columns, then $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$

- preserves inner product:

$$(\boldsymbol{A}\boldsymbol{c})^\top (\boldsymbol{A}\boldsymbol{d})$$

Matrix-Vector Multiplication

If $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ has orthonormal columns, then $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$

- preserves inner product:

$$\begin{aligned} (\boldsymbol{A}\boldsymbol{c})^\top (\boldsymbol{A}\boldsymbol{d}) &= \boldsymbol{c}^\top \boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{d} \\ &= \boldsymbol{c}^\top \boldsymbol{d} \end{aligned}$$

- preserves ℓ_2-norm:

$$\|\boldsymbol{A}\boldsymbol{x}\|_2$$

Matrix-Vector Multiplication

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- preserves inner product:

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- preserves ℓ_2-norm:

$$\begin{aligned}\|\boldsymbol{A}\boldsymbol{x}\|_2 &= \sqrt{(\boldsymbol{A}\boldsymbol{x})^\top\boldsymbol{A}\boldsymbol{x}} \\ &= \sqrt{\boldsymbol{x}^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x}} \\ &= \sqrt{\boldsymbol{x}^\top\boldsymbol{x}} \\ &= \|\boldsymbol{x}\|_2\end{aligned}$$

- preserves angles:

$$\cos(\theta) = \frac{(\boldsymbol{A}\boldsymbol{c})^\top(\boldsymbol{A}\boldsymbol{d})}{\|\boldsymbol{A}\boldsymbol{c}\|_2 \cdot \|\boldsymbol{A}\boldsymbol{d}\|_2}$$

Matrix-Vector Multiplication

If $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ has orthonormal columns, then $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$

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Angle between Two Vectors

For vectors $\boldsymbol{c} = (1, 2, 1)^\top$ and $\boldsymbol{d} = (0, 2, 0)^\top$:

- Angle

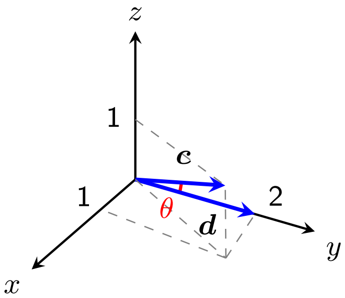
$$\cos(\theta) = \frac{\boldsymbol{c}^\top \boldsymbol{d}}{\|\boldsymbol{c}\|_2 \cdot \|\boldsymbol{d}\|_2} = \frac{\sqrt{6}}{3}$$

- For matrix $\boldsymbol{A} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \end{bmatrix}$,

  we have

$$\boldsymbol{A}\boldsymbol{c} = \left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right)^\top$$

$$\boldsymbol{A}\boldsymbol{d} = \left(\sqrt{2}, \sqrt{2}, 0\right)^\top$$



Angle between Two Vectors

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- Angle

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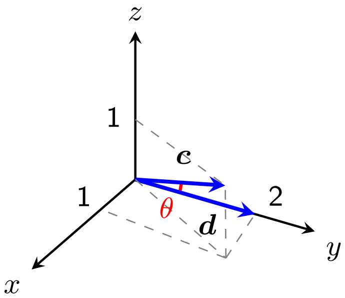
- For matrix $\boldsymbol{A} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \end{bmatrix}$,

  we have

$$\boldsymbol{A}\boldsymbol{c} = \left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right)^\top$$

$$\boldsymbol{A}\boldsymbol{d} = \left(\sqrt{2}, \sqrt{2}, 0\right)^\top$$

$$\Rightarrow \quad \cos(\theta) = \frac{(\boldsymbol{A}\boldsymbol{c})^\top (\boldsymbol{A}\boldsymbol{d})}{\|\boldsymbol{A}\boldsymbol{c}\|_2 \cdot \|\boldsymbol{A}\boldsymbol{d}\|_2} = \frac{4}{\sqrt{6} \times 2}$$



Orthogonal Matrix

Solving linear systems:

- Linear equation with orthogonal coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

solution is

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \boldsymbol{A}^{\top}\boldsymbol{b}$$

Orthogonal Matrix

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- Linear equation with orthogonal coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$:

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  solution is

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \boldsymbol{A}^\top \boldsymbol{b}$$

- **Example**: Solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\boldsymbol{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \Rightarrow \boldsymbol{A}^\top = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

  solution is

$$\boldsymbol{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Quick Summary

Monday's Class:

- Adjoint of 3×3 matrices (minors $\rightarrow$ cofactor matrix $\rightarrow$ transpose of cofactor matrix)
- Inverse of 3×3 matrices
- Orthonormal vectors $\rightarrow$ orthogonal matrix (inner product, ℓ_2-norm, and angles)

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Review: Gauss Elimination

- Problem:

$$\begin{aligned} 3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2 \end{aligned} \quad \Rightarrow \quad [\boldsymbol{A} \mid \boldsymbol{b}] = \left[\begin{array}{cc|c} 3 & 2 & 18 \\ -1 & 2 & 2 \end{array}\right]$$

- **Upper triangular** form:

$$\left[\begin{array}{cc|c} 3 & 2 & 18 \\ 0 & \frac{8}{3} & 8 \end{array}\right]$$

- Solution:

$$x_1 = 4, \qquad x_2 = 3$$

LU Decomposition

- **Concept**: LU decomposition is a method that factors a square matrix $\boldsymbol{A}$ into **two triangular matrices**:
  - A **Lower** triangular matrix $(\boldsymbol{L})$

$$\begin{bmatrix} * & & & \\ * & * & & \\ * & * & * & \\ * & * & * & * \end{bmatrix} \qquad \text{(zeros above the diagonal)}$$

LU Decomposition

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  - A **Lower** triangular matrix ($\boldsymbol{L}$)

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  - An **Upper** triangular matrix ($\boldsymbol{U}$)

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LU Decomposition

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$$\begin{bmatrix} * & & & \\ * & * & & \\ * & * & * & \\ * & * & * & * \end{bmatrix} \quad \text{(zeros above the diagonal)}$$

  - An **Upper** triangular matrix ($\boldsymbol{U}$)

$$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix} \quad \text{(zeros below the diagonal)}$$

- Math expression:

$$\boldsymbol{A} = \boldsymbol{LU}$$

- **Efficiency**: This is particularly useful when solving $\boldsymbol{Ax} = \boldsymbol{b}$ for multiple different right-hand side vectors $\boldsymbol{b}$.

LU Decomposition

If there exists $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, then the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is solved in two distinct steps:

1. **Forward Substitution**: Solve

$$\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$$

for the intermediate vector $\boldsymbol{y}$.

We use a unit lower triangular matrix such that

$$\begin{bmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & \\ * & * & * & 1 \end{bmatrix} \qquad \text{(ones on the diagonal)}$$

LU Decomposition

If there exists $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, then the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is solved in two distinct steps:

1. **Forward Substitution**: Solve

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We use a unit lower triangular matrix such that

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2. **Back Substitution**: Solve

$$\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$$

for the final variables $\boldsymbol{x}$.

LU Decomposition of 2×2 Matrices

Solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$?

- The linear system such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- Assume the lower and upper triangular matrices:

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}, \qquad \boldsymbol{U} = \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix}$$

LU Decomposition of 2×2 Matrices

Solve $\boldsymbol{Ax} = \boldsymbol{b}$?

- The linear system such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- Assume the lower and upper triangular matrices:

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}, \qquad \boldsymbol{U} = \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix}$$

- LU decomposition:

$$\boldsymbol{LU} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \\ \ell \cdot u_1 & \ell \cdot u_2 + u_3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Thus, we have

$$\begin{cases} u_1 = a \\ u_2 = b \\ \ell \cdot u_1 = c \\ \ell \cdot u_2 + u_3 = d \end{cases} \Rightarrow \underbrace{\boldsymbol{L} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} a & b \\ 0 & d - \ell \cdot b \end{bmatrix}}_{\text{How to find the solution } \boldsymbol{x}?}$$

LU Decomposition of 2×2 Matrices

- Problem:

$$\begin{aligned} 3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2 \end{aligned} \quad \Rightarrow \quad \boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$$

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- LU decomposition:

$$\begin{bmatrix} u_1 & u_2 \\ \ell \cdot u_1 & \ell \cdot u_2 + u_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} u_1 = 3 \\ u_2 = 2 \\ \ell \cdot u_1 = -1 \\ \ell \cdot u_2 + u_3 = 2 \end{cases} \quad \Rightarrow \quad \begin{cases} u_1 = 3 \\ u_2 = 2 \\ \ell = -1/3 \\ u_3 = 8/3 \end{cases}$$

Forward & Back Substitutions

- (**Forward**) Solve $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$:

$$\begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solution of immediate variables $\boldsymbol{y}$:

$$y_1 = b_1, \qquad y_2 = b_2 - \ell \cdot y_1$$

Forward & Back Substitutions

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$$\begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

  Solution of immediate variables $\boldsymbol{y}$:

$$y_1 = b_1, \qquad y_2 = b_2 - \ell \cdot y_1$$

- (**Back**) Solve $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$:

$$\begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

  Solution of final variables $\boldsymbol{x}$:

$$x_2 = y_2/u_3, \qquad x_1 = (y_1 - u_2 x_2)/u_1$$

Forward & Back Substitutions

- (**Forward**) Solve $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$:

$$\begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solution of immediate variables $\boldsymbol{y}$:

$$y_1 = b_1, \qquad \frac{c}{a} y_1 + y_2 = b_2 \quad \Rightarrow \quad y_2 = b_2 - \frac{c \cdot b_1}{a}$$

Forward & Back Substitutions

- (**Forward**) Solve $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$:

$$\begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solution of immediate variables $\boldsymbol{y}$:

$$y_1 = b_1, \qquad \frac{c}{a} y_1 + y_2 = b_2 \quad \Rightarrow \quad y_2 = b_2 - \frac{c \cdot b_1}{a}$$

- (**Back**) Solve $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$:

$$\begin{bmatrix} a & b \\ 0 & d - \ell \cdot b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - \frac{c \cdot b_1}{a} \end{bmatrix}$$

Solution of final variables $\boldsymbol{x}$:

$$x_1 = \frac{b_1}{a} - \frac{b(a \cdot b_2 - c \cdot b_1)}{a^2(d - \ell \cdot b)}, \qquad x_2 = \frac{a \cdot b_2 - c \cdot b_1}{a(d - \ell \cdot b)}$$

Forward & Back Substitutions

Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with n variables?

- Let $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ be the LU decomposition of $\boldsymbol{A}$:

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{2,1} & 1 & 0 & \cdots & 0 \\ \ell_{3,1} & \ell_{3,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n,1} & \ell_{n,2} & \ell_{n,3} & \cdots & 1 \end{bmatrix} \qquad \boldsymbol{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & u_{33} & \cdots & u_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{n,n} \end{bmatrix}$$

- **Forward substitution**: Solving the lower triangular system $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$ from top to bottom:

$$\begin{aligned} y_1 &= b_1 \\ y_2 &= b_2 - \ell_{2,1} y_1 \\ &\vdots \\ y_n &= b_n - \ell_{n,1} y_1 - \cdots - \ell_{n,n-1} y_{n-1} \end{aligned}$$

Forward & Back Substitutions

Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with n variables?

- Let $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ be the LU decomposition of $\boldsymbol{A}$:

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- **Back substitution**: Solving the upper triangular system $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$ from bottom to top:

$$\begin{aligned} x_n &= y_n / u_{n,n} \\ x_{n-1} &= (y_{n-1} - u_{n-1,n} x_n) / u_{n-1,n-1} \\ &\vdots \\ x_1 &= (y_1 - u_{1,2} x_2 - \cdots - u_{1,n} x_n) / u_{1,1} \end{aligned}$$

Solving Linear Systems

Solve the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

- Solve the lower triangular system $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$:

$$\begin{aligned} y_1 &= -1 \\ y_2 &= 1 - 2(-1) = 3 \\ y_3 &= 2 - (-1) - 3 = 0 \end{aligned}$$

- Solve the upper triangular system $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$:

$$\begin{aligned} x_3 &= 0 \\ x_2 &= 3 \\ x_1 &= (-1 - 4(3) - 0)/2 = -13/2 \end{aligned}$$

Quick Summary

Wednesday's Class:

- Essential idea of LU decomposition
- LU decomposition for solving $\boldsymbol{Ax} = \boldsymbol{b}$

Solving Linear Systems: Comparison

Solving the linear system with different methods:

$$
\begin{aligned}
3x_1 + 2x_2 &= 18 \\
-x_1 + 2x_2 &= 2
\end{aligned}
$$

1. Graphical method

$$
\begin{aligned}
x_2 &= -\frac{3}{2}x_1 + 9 \\
x_2 &= \frac{1}{2}x_1 + 1
\end{aligned}
$$

2. Cramer's rule:

$$
x_1 = \frac{\det(\boldsymbol{A}_1)}{\det(\boldsymbol{A})}, \qquad x_2 = \frac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})}
$$

3. Inverse of matrix: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} \quad \Rightarrow \quad \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

$$
\boldsymbol{A}^{-1} = \frac{\operatorname{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})} \qquad \text{(adjoint matrix over determinant)}
$$

4. Gauss elimination (including back substitution)
5. LU decomposition (including both forward and back substitutions)

Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

  "Class Participation Quiz 21"

  Time slot: **2:30PM – 3:00PM**

  on Canvas.

Condition Number

- **Motivation**: When solving $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, we assume our data ($\boldsymbol{A}$ and $\boldsymbol{b}$) is perfect.

  In reality, data contains noise, measurement errors, or rounding errors.

- How **stable** is the solution of $\boldsymbol{x}$?

$$\boldsymbol{A}\underbrace{(\boldsymbol{x} + \Delta\boldsymbol{x})}_{\text{output}} = \underbrace{(\boldsymbol{b} + \Delta\boldsymbol{b})}_{\text{input}}$$

If we slightly change $\boldsymbol{b}$ by a small perturbation $\Delta\boldsymbol{b}$, how does the solution of $\boldsymbol{x}$ change? How large is $\Delta\boldsymbol{x}$?

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If we slightly change $\boldsymbol{b}$ by a small perturbation $\Delta\boldsymbol{b}$, how does the solution of $\boldsymbol{x}$ change? How large is $\Delta\boldsymbol{x}$?

- **Problem**: Some systems are **sensitive**. A small change in the input can lead to a massive change in the output.

Condition Number

- **Definition**: The **Condition Number** $\kappa(\boldsymbol{A})$ is a measure of how sensitive a linear system is to these small errors.

$$\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{A}^{-1}\|_F$$

Condition Number

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$$\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{A}^{-1}\|_F$$

  - Frobenius norm of $\boldsymbol{A}$:

$$\|\boldsymbol{A}\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}^2}$$

  - ℓ_2-norm of $\boldsymbol{x}$:

$$\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

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$$\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

- It always holds that $\kappa(\boldsymbol{A}) \geq 1$.

Condition Number: Derivation (Optional)

Prove that $\|\boldsymbol{Ax}\|_2 \leq \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{x}\|_2$.

- Let $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_n$ be the rows of the matrix $\boldsymbol{A}$.

$$[\boldsymbol{Ax}]_i = \boldsymbol{d}_i^\top \boldsymbol{x}$$

Each element is the inner product of the i-th row and $\boldsymbol{x}$.

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Each element is the inner product of the i-th row and $\boldsymbol{x}$.

- By the **Cauchy-Schwarz inequality** for vectors, the absolute value of the inner product is bounded by the product of their ℓ_2-norms:

$$|[\boldsymbol{Ax}]_i| = |\boldsymbol{d}_i^\top \boldsymbol{x}| \leq \|\boldsymbol{d}_i\|_2 \cdot \|\boldsymbol{x}\|_2$$

- Square both sides and sum:

$$\|\boldsymbol{Ax}\|_2^2 = \sum_{i=1}^{n} |\boldsymbol{d}_i^\top \boldsymbol{x}|^2 \leq \left(\sum_{i=1}^{n} \|\boldsymbol{d}_i\|_2^2 \right) \cdot \|\boldsymbol{x}\|_2^2 = \|\boldsymbol{A}\|_F^2 \cdot \|\boldsymbol{x}\|_2^2$$

Condition Number: Derivation (Optional)

Prove that $\|\boldsymbol{A}\boldsymbol{x}\|_2 \leq \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{x}\|_2$.

- Let $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_n$ be the rows of the matrix $\boldsymbol{A}$.

$$[\boldsymbol{A}\boldsymbol{x}]_i = \boldsymbol{d}_i^\top \boldsymbol{x}$$

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- Take the square root of both sides:

$$\|\boldsymbol{A}\boldsymbol{x}\|_2 \leq \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{x}\|_2$$

Condition Number: Derivation

Prove that

$$\frac{\|\Delta \boldsymbol{x}\|_2/\|\boldsymbol{x}\|_2}{\|\Delta \boldsymbol{b}\|_2/\|\boldsymbol{b}\|_2} \leq \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{A}^{-1}\|_F = \kappa(\boldsymbol{A})$$

- From the linear system $\boldsymbol{A}(\boldsymbol{x} + \underbrace{\Delta \boldsymbol{x}}_{\text{error } \hat{\boldsymbol{x}} - \boldsymbol{x}}) = (\boldsymbol{b} + \Delta \boldsymbol{b})$, we have

$$\boldsymbol{A}\Delta \boldsymbol{x} = \Delta \boldsymbol{b}$$

- How the relative error in output $\|\Delta \boldsymbol{x}\|_2/\|\boldsymbol{x}\|_2$ is related to the relative error in the input $\|\Delta \boldsymbol{b}\|_2/\|\boldsymbol{b}\|_2$?

Condition Number: Derivation

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- How the relative error in output $\|\Delta \boldsymbol{x}\|_2/\|\boldsymbol{x}\|_2$ is related to the relative error in the input $\|\Delta \boldsymbol{b}\|_2/\|\boldsymbol{b}\|_2$?

$$\begin{aligned}
\frac{\|\Delta \boldsymbol{x}\|_2/\|\boldsymbol{x}\|_2}{\|\Delta \boldsymbol{b}\|_2/\|\boldsymbol{b}\|_2} &= \frac{\|\Delta \boldsymbol{x}\|_2 \cdot \|\boldsymbol{b}\|_2}{\|\Delta \boldsymbol{b}\|_2 \cdot \|\boldsymbol{x}\|_2} \\
&= \frac{\|\boldsymbol{A}^{-1}\Delta \boldsymbol{b}\|_2 \cdot \|\boldsymbol{A}\boldsymbol{x}\|_2}{\|\Delta \boldsymbol{b}\|_2 \cdot \|\boldsymbol{x}\|_2} \\
&\leq \frac{\|\boldsymbol{A}^{-1}\|_F \cdot \|\Delta \boldsymbol{b}\|_2 \cdot \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{x}\|_2}{\|\Delta \boldsymbol{b}\|_2 \cdot \|\boldsymbol{x}\|_2} \\
&= \|\boldsymbol{A}^{-1}\|_F \cdot \|\boldsymbol{A}\|_F = \kappa(\boldsymbol{A})
\end{aligned}$$

Error Bound

- Condition number:

$$\frac{\|\Delta \boldsymbol{x}\|_2 / \|\boldsymbol{x}\|_2}{\|\Delta \boldsymbol{b}\|_2 / \|\boldsymbol{b}\|_2} \leq \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{A}^{-1}\|_F = \kappa(\boldsymbol{A})$$

- Error bound:

$$\underbrace{\frac{\|\Delta \boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2}}_{\text{Output error}} \leq \kappa(\boldsymbol{A}) \underbrace{\frac{\|\Delta \boldsymbol{b}\|_2}{\|\boldsymbol{b}\|_2}}_{\text{Input error}}$$

  - **Input error**: The noise or uncertainty in your starting data.
  - **Condition number**: A multiplier (the sensitivity of the system).
  - **Output error**: The maximum possible inaccuracy in your result.

Condition Number: Description

The condition number measures how sensitive a mathematical problem is to small changes in input data (such as measurement errors or rounding).

Condition Number	**Description**	**Sensitivity**
Small (Close to 1)	**Well-Conditioned**	Small changes in input result in small changes in output. The solution is reliable.
Large $(10^3, 10^6, \dots)$	**Ill-Conditioned**	Small changes in input can cause remarkable changes in output. The solution is unstable.

- In structural analysis, ill-conditioning often occurs in models with vastly different stiffnesses.

Ill-Conditioned Systems

- Problem:

$$x_1 + 2x_2 = 10$$
$$1.1x_1 + 2x_2 = 10.4$$

- Exact solution: Solving this system yields $x_1 = 4$ and $x_2 = 3$.

Ill-Conditioned Systems

- Problem:

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$$1.1x_1 + 2x_2 = 10.4$$

- Exact solution: Solving this system yields $x_1 = 4$ and $x_2 = 3$.
- What happens with a small change in $\boldsymbol{b}$?
  - Let's change the right-hand side constant b_2 from 10.4 to 10.8:

$$x_1 + 2x_2 = 10$$
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    - Let's change the right-hand side constant b_2 from 10.4 to 10.8:

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$$1.1x_1 + 2x_2 = 10.8$$

    - Subtract the equations: $0.1x_1 = 0.8 \Rightarrow x_1 = 8$
    - Substitute back: $8 + 2x_2 = 10 \Rightarrow x_2 = 1$
- **Result**: A small change in the input vector $\boldsymbol{b}$ caused the solution to jump from $(4, 3)$ to $(8, 1)$.
- **Small change in $\boldsymbol{b} \rightarrow$ remarkable change in $\boldsymbol{x}$**

Ill-Conditioned Systems

Why is this happening?

- For matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 1.1 & 2 \end{bmatrix}$
  - Frobenius norm of $\boldsymbol{A}$:

$$\|\boldsymbol{A}\|_F = \sqrt{1^2 + 2^2 + 1.1^2 + 2^2} = \sqrt{10.21} \approx 3.195$$

  - Inverse of $\boldsymbol{A}$:

$$\boldsymbol{A}^{-1} = \frac{\mathrm{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})} = \frac{1}{-0.2} \begin{bmatrix} 2 & -2 \\ -1.1 & 1 \end{bmatrix} = \begin{bmatrix} -10 & 10 \\ 5.5 & -5 \end{bmatrix}$$

  - Frobenius norm of $\boldsymbol{A}^{-1}$:

$$\|\boldsymbol{A}^{-1}\|_F = \sqrt{(-10)^2 + 10^2 + 5.5^2 + (-5)^2} = \sqrt{255.25} \approx 15.977$$

Ill-Conditioned Systems

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- For matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 1.1 & 2 \end{bmatrix}$
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  $$\|\boldsymbol{A}^{-1}\|_F = \sqrt{(-10)^2 + 10^2 + 5.5^2 + (-5)^2} = \sqrt{255.25} \approx 15.977$$
- Condition number:
$$\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{A}^{-1}\|_F \approx 51.05$$

Ill-Conditioned Systems

Geometric Perspective:

- Geometrically, solving a 2×2 system is finding the intersection of two lines.
- Equation 1:

$$x_2 = \underbrace{-0.5}_{\text{slope}} x_1 + 5$$

- Equation 2:

$$x_2 = \underbrace{-0.55}_{\text{slope}} x_1 + 5.2$$

- The lines are nearly parallel.
- A very small shift in one line moves the intersection point a great distance along the other line. This is the intuition of an ill-conditioned system.

Well-Conditioned Systems

- Problem:

$$
\begin{aligned}
3x_1 + 2x_2 &= 18 \\
-x_1 + 2x_2 &= 2
\end{aligned}
$$

- Exact solution: Solving this system yields $x_1 = 4$ and $x_2 = 3$.

Well-Conditioned Systems

- Problem:

$$\begin{aligned} 3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2 \end{aligned}$$

- Exact solution: Solving this system yields $x_1 = 4$ and $x_2 = 3$.
- What happens with a small change in $\boldsymbol{b}$?
    - Let's introduce a small change to the right-hand side. Change b_2 from 2 to 2.4:

$$\begin{aligned} 3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2.4 \end{aligned}$$

Well-Conditioned Systems

- Problem:

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

- Exact solution: Solving this system yields $x_1 = 4$ and $x_2 = 3$.
- What happens with a small change in $\boldsymbol{b}$?
  - Let's introduce a small change to the right-hand side. Change b_2 from 2 to 2.4:

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2.4$$

  - Subtract the two equations: $4x_1 = 15.6 \Rightarrow x_1 = 3.9$
  - Substituting back: $-3.9 + 2x_2 = 2.4 \Rightarrow 2x_2 = 6.3 \Rightarrow x_2 = 3.15$
- **Small change in $\boldsymbol{b} \rightarrow$ small change in $\boldsymbol{x}$**

Well-Conditioned Systems

Why is this happening?

- For matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$

Well-Conditioned Systems

Why is this happening?

- For matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$
  - Frobenius norm of $\boldsymbol{A}$:

$$\|\boldsymbol{A}\|_F = \sqrt{3^2 + 2^2 + (-1)^2 + 2^2} = 3\sqrt{2}$$

  - Inverse of $\boldsymbol{A}$:

$$\boldsymbol{A}^{-1} = \frac{\mathrm{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})} = \frac{1}{8}\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$$

  - Frobenius norm of $\boldsymbol{A}^{-1}$:

$$\|\boldsymbol{A}^{-1}\|_F = \frac{3\sqrt{2}}{8}$$

Well-Conditioned Systems

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- For matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$
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  - Frobenius norm of $\boldsymbol{A}^{-1}$:

$$\|\boldsymbol{A}^{-1}\|_F = \frac{3\sqrt{2}}{8}$$

- Condition number:

$$\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{A}^{-1}\|_F = \frac{9}{4} = 2.25$$

- Interpretation: Since the condition number is very close to 1, the system is well-conditioned.

Quick Summary

Friday's Class:

- Definition of condition number
- Derivation of condition number
- Ill-conditioned system vs. well-conditioned system