Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 8: Introduction to Applied Linear Algebra: Part I

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Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

  "`Class Participation Quiz 16`"

  Time slot: **2:30PM – 3:00PM**

  on Canvas.

Linear Systems

- A system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

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- A system of n linear algebraic equations:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i$$

for $i = 1, 2, \ldots, n$.

Linear Systems

- A system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

- A system of n linear algebraic equations:

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for $i = 1, 2, \ldots, n$.

- Matrix form $\boldsymbol{Ax} = \boldsymbol{b}$
- **Focus**: The existence and uniqueness of solutions

Structural Stiffness Analogy

In civil engineering:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$$

- $\boldsymbol{K}$ is the global stiffness matrix
- $\boldsymbol{u}$ is the displacement vector (unknowns)
- $\boldsymbol{F}$ is applied nodal loads

Matrix Equation

In the field of control theory, how to find unique solution to $\boldsymbol{X}$?

- A Lyapunov equation is a **matrix equation** of the form:

$$\boldsymbol{A}^\top \boldsymbol{X} + \boldsymbol{X}\boldsymbol{A} = \boldsymbol{C}$$

  for any $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{C} \in \mathbb{R}^{n \times n}$

- A Sylvester equation is a **matrix equation** of the form:

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{B} = \boldsymbol{C}$$

  for any $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, $\boldsymbol{B} \in \mathbb{R}^{n \times n}$, and $\boldsymbol{C} \in \mathbb{R}^{m \times n}$

Linear Transformation

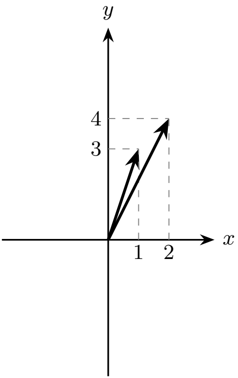
- **Definition**: Just as a linear function $f(x) = ax$ scales a number x, a linear transformation $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ maps a vector $\boldsymbol{x} \in \mathbb{R}^n$ to another vector $\boldsymbol{A}\boldsymbol{x} \in \mathbb{R}^n$.
- **Key Difference**: In a linear transformation, the input and output are vectors rather than single scalar numbers.
- **Mechanism**: The transformation occurs by moving the vector $\boldsymbol{x}$ to a new position $\boldsymbol{A}\boldsymbol{x}$ through the operation of matrix $\boldsymbol{A}$.

Linear Transformation

To understand $\boldsymbol{Ax}$ geometrically, consider $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

1. **Identify Basis Vectors**: Extract the column vectors of matrix $\boldsymbol{A}$:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



Linear Transformation

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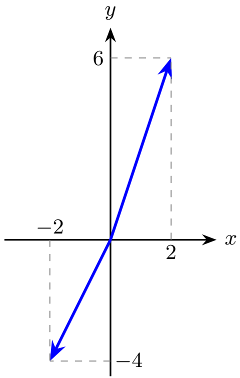
❷ **Scale the Basis**: Multiply each basis vector by 2 and -1:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \times 2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \times (-1) = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

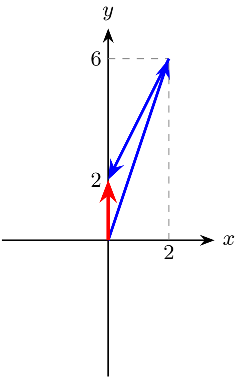


Linear Transformation

To understand $\boldsymbol{Ax}$ geometrically, consider $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

❸ **Vector Addition**: Sum the scaled vectors to find the transformed vector:

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

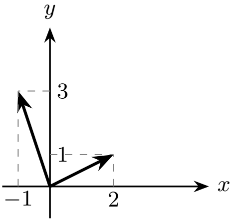


Linear Transformation

Understand Ax geometrically

- Consider $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$, we have

$$\boldsymbol{Ax} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = 2 \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{2} \times \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ \frac{7}{2} \end{bmatrix}$$



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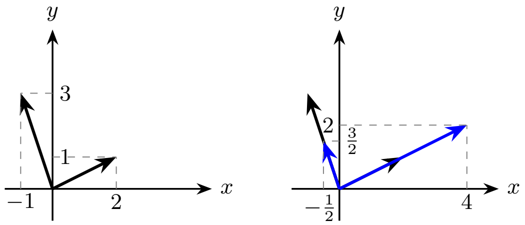

Linear Transformation

Understand Ax geometrically

- Consider $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$, we have

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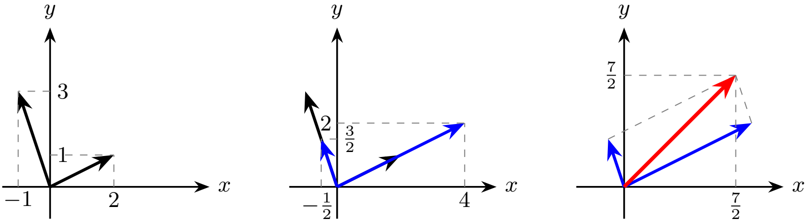

Matrix Multiplication

- If $\boldsymbol{C} = \boldsymbol{AB}$, then the element c_{ij} is defined by the inner product:

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

- **Example**: Define two matrices:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Matrix Multiplication

- If $\boldsymbol{C} = \boldsymbol{AB}$, then the element c_{ij} is defined by the inner product:

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- **Example**: Define two matrices:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

  To find each element c_{ij}, we take the **inner product** of the i-th row of $\boldsymbol{A}$ and the j-th column of $\boldsymbol{B}$:

  - For c_{11}: $(1 \times 5) + (2 \times 7) = 5 + 14 = \mathbf{19}$
  - For c_{12}: $(1 \times 6) + (2 \times 8) = 6 + 16 = \mathbf{22}$
  - For c_{21}: $(3 \times 5) + (4 \times 7) = 15 + 28 = \mathbf{43}$
  - For c_{22}: $(3 \times 6) + (4 \times 8) = 18 + 32 = \mathbf{50}$

- Python programming: `C = A @ B`

Matrix Multiplication

- **Complexity**: $\mathcal{O}(n^3)$ operations for two $n \times n$ matrices
- Why is it $\mathcal{O}(n^3)$?
  - There are n^2 total elements in the resulting matrix $(2 \times 2 = 4)$
  - To calculate each element, we perform n multiplications (2 in this case)
  - Total operations $\approx n^2 \times n = n^3$

Matrix Multiplication

Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{X} = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$, compute $\boldsymbol{AX}$.

Matrix Multiplication

Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{X} = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$, compute $\boldsymbol{AX}$.

$$\begin{aligned} \boldsymbol{AX} &= \begin{bmatrix} 1\times 5 + 2\times 8 & 1\times 6 + 2\times 9 & 1\times 7 + 2\times 10 \\ 3\times 5 + 4\times 8 & 3\times 6 + 4\times 9 & 3\times 7 + 4\times 10 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{bmatrix} \end{aligned}$$

Determinant of Matrix

- **Definition**: The determinant is a scalar value that can be computed from the elements of a square matrix, i.e., $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$
- **Geometric Significance**: It represents the volume scaling factor of the linear transformation described by the matrix
- **Singularity**: If the determinant is zero, the matrix is singular, meaning the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ may have no unique solution
- **Python**: `np.linalg.det(A)`

Determinant of Matrix

Matrix of size 2×2:

- The determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Determinant of Matrix

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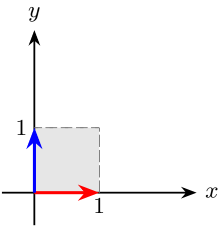
- The determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

- Intuitive understanding of $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$
    - Draw column vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the area is 1



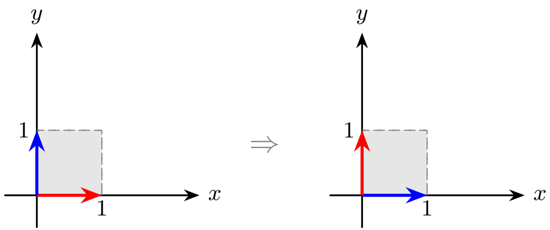
Determinant of Matrix

Matrix of size 2×2:

- The determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

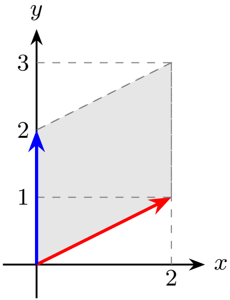
$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

- Intuitive understanding of $\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0 - 1 = -1$
  - Draw column vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the area is 1 (reversed direction)



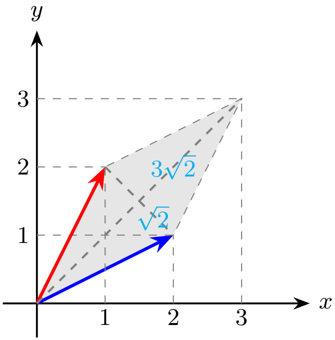
Determinant of Matrix

- Intuitive understanding of the determinant of $\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$
  - Draw column vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$
  - Area is 4, so $\det(\boldsymbol{A}) = 4$



Determinant of Matrix

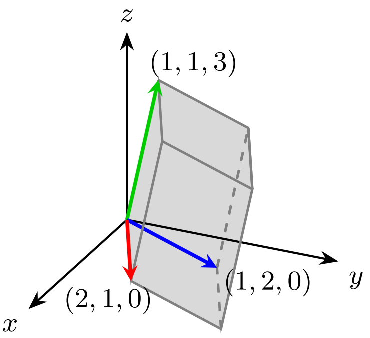
- Intuitive understanding of the determinant of $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
  - Draw column vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
  - Area is $\frac{1}{2} \times 3\sqrt{2} \times \sqrt{2} = 3$ (reversed direction), so $\det(\boldsymbol{A}) = -3$



Determinant of Matrix

Matrix of size 3×3:

- Intuitive understanding of the determinant of $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
  - Draw column vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, volume $\det(\boldsymbol{A}) = 9$



Quick Summary

Monday's Class:

- A system of linear equations
- Linear transformation
- Matrix multiplication
- Determinant of square matrices

Quizzes Now!

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  on Canvas.

General Form of Linear Systems

- A set of n linear equations with n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

  - a_{ij}: constant coefficients
  - b_i: constants (right-hand side)
  - x_j: variables to be solved
- The system can be written compactly as $\boldsymbol{Ax} = \boldsymbol{b}$ such that

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Methods for Small Systems ($n \leq 3$)

For small systems, we can use:

- **Graphical method** (2 equations only)
- **Cramer's rule** (using determinants)
- **Elimination** of variables

Graphical Method

- For two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Solve each for x_2:

$$x_2 = -\frac{a_{11}}{a_{12}}x_1 + \frac{b_1}{a_{12}}$$
$$x_2 = -\frac{a_{21}}{a_{22}}x_1 + \frac{b_2}{a_{22}}$$

- **Intersection** is the solution.

Graphical Method

- Problem:

$$\begin{aligned} 3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2 \end{aligned}$$

- Solve each for x_2:

$$x_2 = -\frac{3}{2}x_1 + 9$$

$$x_2 = \frac{1}{2}x_1 + 1$$

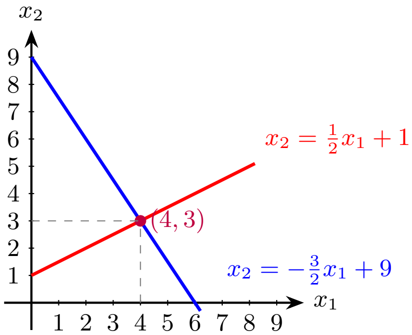
Graphical Method

- Problem:

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

- Solve each for x_2:

$$x_2 = -\frac{3}{2}x_1 + 9$$
$$x_2 = \frac{1}{2}x_1 + 1$$



Determinants of 2×2 or 3×3 Matrices

- The determinant of $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(\boldsymbol{A}) = ad - bc$$

Determinants of 2×2 or 3×3 Matrices

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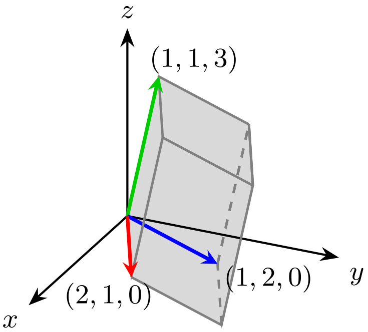
- The determinant of $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is

$$\det(\boldsymbol{A}) = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \cdot \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

Determinants of 2×2 or 3×3 Matrices

Compute the determinant of $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$?

$$\begin{aligned} \det(\boldsymbol{A}) &= 2 \times \det\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\right) - 1 \times \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}\right) + 1 \times \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) \\ &= 2 \times 6 - 1 \times 3 + 0 = 9 \end{aligned}$$



Determinants of 2×2 or 3×3 Matrices

Compute the determinant of $\boldsymbol{A} = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$?

Cramer's Rule

- Each variable is a fraction of two determinants:

$$x_i = \frac{\det(\boldsymbol{A}_i)}{\det(\boldsymbol{A})}$$

where $\boldsymbol{A}_i$ is matrix $\boldsymbol{A}$ with column i replaced by the right-hand side vector $\boldsymbol{b}$.

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where $\boldsymbol{A}_i$ is matrix $\boldsymbol{A}$ with column i replaced by the right-hand side vector $\boldsymbol{b}$.

- For two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Solution:

$$x_1 = \frac{\det(\boldsymbol{A}_1)}{\det(\boldsymbol{A})} = \frac{\det\left(\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)} = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})} = \frac{\det\left(\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)} = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

Cramer's Rule

- Problem:

$$\begin{aligned} 3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2 \end{aligned} \qquad \Rightarrow \qquad \boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 18 \\ 2 \end{bmatrix}$$

- Determinants:

$$\det(\boldsymbol{A}) = 8$$
$$\det(\boldsymbol{A}_1) = \det\left(\begin{bmatrix} 18 & 2 \\ 2 & 2 \end{bmatrix}\right) = 32$$
$$\det(\boldsymbol{A}_2) = \det\left(\begin{bmatrix} 3 & 18 \\ -1 & 2 \end{bmatrix}\right) = 24$$

Cramer's Rule

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- Determinants:

$$\det(\boldsymbol{A}) = 8$$

$$\det(\boldsymbol{A}_1) = \det\left(\begin{bmatrix} 18 & 2 \\ 2 & 2 \end{bmatrix}\right) = 32$$

$$\det(\boldsymbol{A}_2) = \det\left(\begin{bmatrix} 3 & 18 \\ -1 & 2 \end{bmatrix}\right) = 24$$

- Solution: $x_1 = \dfrac{\det(\boldsymbol{A}_1)}{\det(\boldsymbol{A})} = 4$ and $x_2 = \dfrac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})} = 3.$

Cramer's Rule

- For two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Basic strategy:
  - Multiply equations by constants:

$$a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = a_{21}b_1 \qquad \text{(multiply by } a_{21}\text{)}$$
$$a_{21}a_{11}x_1 + a_{22}a_{11}x_2 = a_{11}b_2 \qquad \text{(multiply by } a_{11}\text{)}$$

Cramer's Rule

- For two equations:

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- Basic strategy:
  - Multiply equations by constants:

$$a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = a_{21}b_1 \qquad \text{(multiply by } a_{21}\text{)}$$
$$a_{21}a_{11}x_1 + a_{22}a_{11}x_2 = a_{11}b_2 \qquad \text{(multiply by } a_{11}\text{)}$$

  - Subtract to eliminate x_1:

$$(a_{22}a_{11} - a_{12}a_{21})x_2 = a_{11}b_2 - a_{21}b_1$$

  - Solution:

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{22}a_{11} - a_{12}a_{21}} \quad \Rightarrow \quad x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

Gauss Elimination

- Problem:

$$\begin{aligned} 8x_1 + 2x_2 - 2x_3 &= -2 \\ 10x_1 + 2x_2 + 4x_3 &= 4 \\ 12x_1 + 2x_2 + 2x_3 &= 6 \end{aligned}$$

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- Multiple the 1st equation by $\frac{10}{8} = 1.25$:

$$10x_1 + 2.5x_2 - 2.5x_3 = -2.5$$

- Subtract this from the 2nd equation:

$$-0.5x_2 + 6.5x_3 = 6.5$$

Gauss Elimination

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$$-0.5x_2 + 6.5x_3 = 6.5$$

- Multiple the 1st equation by $\frac{12}{8} = 1.5$:

$$12x_1 + 3x_2 - 3x_3 = -3$$

- Subtract this from the 3rd equation:

$$-x_2 + 5x_3 = 9$$

Gauss Elimination

- Now, we have

$$\begin{aligned} 8x_1 + 2x_2 - 2x_3 &= -2 \\ 10x_1 + 2x_2 + 4x_3 &= 4 \\ 12x_1 + 2x_2 + 2x_3 &= 6 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 8x_1 + 2x_2 - 2x_3 &= -2 \\ -0.5x_2 + 6.5x_3 &= 6.5 \\ -x_2 + 5x_3 &= 9 \end{aligned}$$

- Multiple the 2nd equation by $\frac{-1}{-0.5} = 2$:

$$-x_2 + 13x_3 = 13$$

- Subtract this from the 3rd equation:

$$-8x_3 = -4$$

- The problem becomes

$$\begin{aligned} 8x_1 + 2x_2 - 2x_3 &= -2 \\ -0.5x_2 + 6.5x_3 &= 6.5 \\ -8x_3 &= -4 \end{aligned} \quad \Rightarrow \quad \begin{cases} x_1 = 1.5 \\ x_2 = -6.5 \\ x_3 = 0.5 \end{cases}$$

Gauss Elimination

Write the system in the form of augmented matrix:

$$\begin{aligned} 8x_1 + 2x_2 - 2x_3 &= -2 \\ 10x_1 + 2x_2 + 4x_3 &= 4 \\ 12x_1 + 2x_2 + 2x_3 &= 6 \end{aligned} \quad \Rightarrow \quad [\boldsymbol{A} \mid \boldsymbol{b}] = \left[\begin{array}{ccc|c} 8 & 2 & -2 & -2 \\ 10 & 2 & 4 & 4 \\ 12 & 2 & 2 & 6 \end{array}\right]$$

- System after first elimination:

$$\left[\begin{array}{ccc|c} 8 & 2 & -2 & -2 \\ 0 & -0.5 & 6.5 & 6.5 \\ 0 & -1 & 5 & 9 \end{array}\right]$$

Now x_1 is eliminated from Rows 2 and 3.

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  Now x_1 is eliminated from Rows 2 and 3.

- **Upper triangular** form:

$$\left[\begin{array}{ccc|c} 8 & 2 & -2 & -2 \\ 0 & -0.5 & 6.5 & 6.5 \\ 0 & 0 & -8 & -4 \end{array}\right]$$

  Now we can solve by **back substitution**.

Gauss Elimination

- A set of n linear equations with n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

- Multiple the first equation by $\frac{a_{21}}{a_{11}}$:

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \cdots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

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$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \cdots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

- Subtract this from the second equation:

$$\underbrace{\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)}_{= a'_{22}} x_2 + \cdots + \underbrace{\left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)}_{= a'_{2n}} x_n = \underbrace{b_2 - \frac{a_{21}}{a_{11}}b_1}_{= b'_2}$$

- Denote new coefficients as $a'_{22}, \ldots, a'_{2n}$ and b'_2

Gauss Elimination

Continue elimination:

- Repeat until the system is **upper triangular**:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

- **Back substitution**:

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}, \qquad x_{n-1} = \frac{b_{n-1}^{(n-2)} - a_{n-1,n}^{(n-2)} x_n}{a_{n-1,n-1}^{(n-2)}}$$

- In general:

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}$$

Gauss Elimination

- Problem:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 1 \\ 5x_1 + 2x_2 + 2x_3 &= -4 \\ 3x_1 + x_2 + x_3 &= 5 \end{aligned} \quad \Rightarrow \quad [\boldsymbol{A} \mid \boldsymbol{b}] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 5 & 2 & 2 & -4 \\ 3 & 1 & 1 & 5 \end{array}\right]$$

Gauss Elimination

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- System after first elimination:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -0.5 & 4.5 & -6.5 \\ 0 & -0.5 & 2.5 & 3.5 \end{array}\right]$$

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- Upper triangular form:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & -0.5 & 4.5 & -6.5 \\ 0 & 0 & -2 & 10 \end{array}\right]$$

- Solution:

$$x_1 = 14, \qquad x_2 = -32, \qquad x_3 = -5$$

Quick Summary

Wednesday's Class:

- **Small systems** can be solved graphically, by Cramer's rule, or by elimination
- **Determinants** indicate singularity (zero) or ill-conditioning (near zero)
- **Gauss elimination** extends elimination to n equations
- **Two phases**: forward elimination $\rightarrow$ upper triangular, then back substitution

Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

"`Class Participation Quiz 18`"

Time slot: **2:30PM – 3:00PM**

on Canvas.

Singular Matrix

- A matrix is **singular** if the determinant is zero, i.e., $\det(\boldsymbol{A}) = 0$
- Meaning: **No unique solution** for $\boldsymbol{Ax} = \boldsymbol{b}$

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- Example:

$$3x_1 + 2x_2 = 18$$
$$6x_1 + 4x_2 = 10$$

  - Determinant: $\det\left(\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}\right) = 3 \times 4 - 2 \times 6 = 0$

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  - Determinant: $\det\left(\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}\right) = 3 \times 4 - 2 \times 6 = 0$
  - Graphical method:

$$x_2 = -\frac{3}{2}x_1 + 9$$
$$x_2 = -\frac{3}{2}x_1 + \frac{5}{2}$$

  No intersection $\rightarrow$ No solution!

Singular Matrix

3×3 matrix:

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- Determinant:

$$\det(\boldsymbol{A}) = 1 \times (2 - 1) - 0 \times (0 - 1) + 1 \times (0 - 1) = 1 - 1 = 0$$

- Geometric result: The three vectors lie on the same two-dimensional plane. The volume they span is zero.

Identify Matrix

Some examples of identity matrix $\boldsymbol{I}$:

$$\underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{2\,\times\,2\ \text{matrix}} \qquad \underbrace{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}_{3\,\times\,3\ \text{matrix}} \qquad \underbrace{\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}}_{4\,\times\,4\ \text{matrix}} \qquad \cdots$$

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Identify Matrix

Multiplication of identity matrix $\boldsymbol{I}$:

- Between identify matrix and any matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

- Between identity matrix and any vector:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Inverse of Matrix

- **Analogy**: Just as the reciprocal of a number a is $1/a$ (where $a \cdot a^{-1} = 1$), a square matrix $\boldsymbol{A}$ may have an inverse $\boldsymbol{A}^{-1}$.
- **Core property**: $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.

$$\text{Identify matrix:} \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

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$$\text{Identify matrix:} \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

- **Requirement**: The inverse $\boldsymbol{A}^{-1}$ exists **if and only if** the matrix is non-singular, meaning $\det(\boldsymbol{A}) \neq 0$.
- If $\det(\boldsymbol{A}) = 0$, the matrix is singular and has no inverse.

Inverse of Matrix

The inverse of 2×2 matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

1. Calculate the determinant:

$$\det(\boldsymbol{A}) = ad - bc$$

2. Construct the adjoint matrix:

$$\text{adj}(\boldsymbol{A}) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Swap the main diagonal elements (a and d)
- Change the signs of the off-diagonal elements (b and c)

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   - Change the signs of the off-diagonal elements (b and c)

3. Divide the entire result by the determinant:

$$\boldsymbol{A}^{-1} = \frac{\operatorname{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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- Given the matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$:
  - The determinant $\det(\boldsymbol{A}) = 3 \times 2 - 2 \times (-1) = 6 + 2 = 8$
  - The adjoint matrix:

$$\text{adj}(\boldsymbol{A}) = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$$

  - The inverse of matrix:

$$\boldsymbol{A}^{-1} = \frac{\text{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})} = \frac{1}{8} \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

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  - Verify $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$?

Inverse of Matrix

```python
import numpy as np

A = np.array([[3, 2], [-1, 2]])
A_inverse = np.linalg.inv(A)
print(A_inverse)
```

Solving Linear Systems

For the linear system $\boldsymbol{Ax} = \boldsymbol{b}$:

- Multiple both sides by the inverse $\boldsymbol{A}^{-1}$:

$$\boldsymbol{A}^{-1}\boldsymbol{Ax} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

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- Core property: $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$ (identify matrix):

$$\boldsymbol{Ix} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

Solving Linear Systems

For the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$:

- Multiple both sides by the inverse $\boldsymbol{A}^{-1}$:

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- Core property: $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$ (identify matrix):

$$\boldsymbol{I}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

- The solution to $\boldsymbol{x}$:

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

- Applications:
  - **Structural Engineering**: Calculating displacements $\boldsymbol{u}$ in the stiffness equation $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$ where $\boldsymbol{u} = \boldsymbol{K}^{-1}\boldsymbol{F}$.
  - **Control Theory**: Solving complex matrix equations, e.g., Lyapunov or Sylvester equations.

Solving Linear Equations

- Problem:

$$\begin{aligned} 3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2 \end{aligned} \quad \Rightarrow \quad \boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 18 \\ 2 \end{bmatrix}$$

- The inverse of matrix $\boldsymbol{A}$:

$$\boldsymbol{A}^{-1} = \frac{\text{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})} = \frac{1}{8}\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

- The solution to $\boldsymbol{x}$:

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 18 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Solving Linear Equations

- Problem:

$$\begin{aligned} 4x_1 - 8x_2 &= -24 \\ -x_1 + 6x_2 &= 34 \end{aligned} \quad \Rightarrow \quad \boldsymbol{A} = \begin{bmatrix} 4 & -8 \\ -1 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -24 \\ 34 \end{bmatrix}$$

Solving Linear Equations

- Problem:

$$\begin{aligned} 4x_1 - 8x_2 &= -24 \\ -x_1 + 6x_2 &= 34 \end{aligned} \quad \Rightarrow \quad \boldsymbol{A} = \begin{bmatrix} 4 & -8 \\ -1 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -24 \\ 34 \end{bmatrix}$$

- Determinant:

$$\det(\boldsymbol{A}) = 4 \times 6 - (-8) \times (-1) = 24 - 8 = 16$$

- Adjoint matrix:

$$\operatorname{adj}(\boldsymbol{A}) = \begin{bmatrix} 6 & 8 \\ 1 & 4 \end{bmatrix}$$

- Inverse of matrix:

Solving Linear Equations

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$$\operatorname{adj}(\boldsymbol{A}) = \begin{bmatrix} 6 & 8 \\ 1 & 4 \end{bmatrix}$$

- Inverse of matrix:

$$\boldsymbol{A}^{-1} = \frac{\operatorname{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})} = \begin{bmatrix} \frac{6}{16} & \frac{8}{16} \\ \frac{1}{16} & \frac{4}{16} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} \\ \frac{1}{16} & \frac{1}{4} \end{bmatrix}$$

- Solution:

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} \\ \frac{1}{16} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} -24 \\ 34 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

Quick Summary

Friday's Class:

- What is the implication of singular matrix?
- The definition of inverse of matrix (e.g., 2×2 matrix)
- Use the inverse of matrix to solve linear systems