Applied Numerical Methods for Civil Engineering

Week 7: Nonlinear Equations

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Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

  "Class Participation Quiz 13"

 Time slot: **2:30PM – 3:00PM**

  on Canvas.

Learning Objectives

By the end of week, you will be able to:

- Derive the Newton-Raphson iteration formula from a Taylor series
- Apply the method to find roots of nonlinear equations
- Analyze the convergence behavior of the method
- Identify situations where the method may fail
- Implement the algorithm with Python programming

Problem Statement

- A nonlinear equation $f(x) = 0$

- Examples:
  - $x^2 - 2 = 0$ (Find $\pm\sqrt{2}$)
  - $\sin(x) = 0$ (Find multiples of π)
  - $e^{-x} - x = 0$ (Intersection of functions)

- Many engineering problems reduce to finding roots!

$f(x) = x^4 - 1.5$
$f(x) = x^3 - 1.5$
$f(x) = x^2 - 1.5$
$f(x) = x - 1.5$
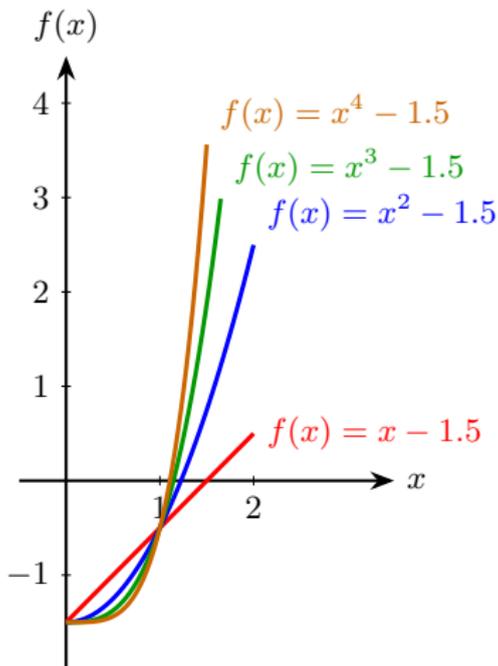
Historical Development

The modern method is a synthesis of three major mathematical contributions:

- **Isaac Newton (1669)**: Described method for polynomial equations
- **Joseph Raphson (1690)**: Formalized the iterative process
- **Thomas Simpson (1740)**: Generalized to calculus form using derivatives

Derivation from Taylor Series

- Expand $f(x)$ around x_n using Taylor series:

$$
\begin{aligned}
f(x_{n+1}) \approx & f(x_n) \\
& + f'(x_n)(x_{n+1} - x_n) \\
& + \frac{f''(x_n)}{2!}(x_{n+1} - x_n)^2 \\
& + \cdots \\
& + \frac{f^{(t)}(x_n)}{t!}(x_{n+1} - x_n)^t \\
= & \sum_{k=0}^{t} \frac{f^{(k)}(x_n)}{k!}(x_{n+1} - x_n)^k
\end{aligned}
$$

Derivation from Taylor Series

- Expand $f(x)$ around x_n using Taylor series:

$$
\begin{aligned}
f(x_{n+1}) \approx & f(x_n) \\
& + f'(x_n)(x_{n+1} - x_n) \\
& + \frac{f''(x_n)}{2!}(x_{n+1} - x_n)^2 \\
& + \cdots \\
& + \frac{f^{(t)}(x_n)}{t!}(x_{n+1} - x_n)^t \\
= & \sum_{k=0}^{t} \frac{f^{(k)}(x_n)}{k!}(x_{n+1} - x_n)^k
\end{aligned}
$$

- We want $f(x_{n+1}) = 0$

- Only use the first-order approximation:

$$
f(x_{n+1}) = 0 \quad \Rightarrow \quad f(x_n) + f'(x_n)(x_{n+1} - x_n) \approx 0
$$

Newton-Raphson Iteration Formula

- Let the first-order approximation be 0:

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0$$

- Rearrange the equation to solve for x_{n+1}:

$$f'(x_n)(x_{n+1} - x_n) = -f(x_n)$$

$$\Rightarrow \quad x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which is the Newton-Raphson iteration formula.

Newton-Raphson Iteration Formula

Understanding the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- x_n: Current estimate
- $f(x_n)$: Function value at current estimate
- $f'(x_n)$: Slope (first-order derivative) at current estimate
- $\frac{f(x_n)}{f'(x_n)}$: Correction term

Intuition: If $f(x_n)$ is large, we take a big step. If slope is steep, we take a smaller step.

Newton-Raphson Method

- Choose initial point x_0 and tolerance ε
- Set $n = 0$
- Repeat until convergence (using `while`)
    - Compute $f(x_n)$ and $f'(x_n)$
    - Calculate new estimate:

    $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

    - Check convergence: If $\frac{|x_{n+1}-x_n|}{|x_{n+1}|} < \varepsilon$, stop
    - Set $n = n + 1$

Stopping Criteria

Three common stopping criteria:

- Absolute error

$$|x_{n+1} - x_n| < \varepsilon$$

- **Relative error**

$$\frac{|x_{n+1} - x_n|}{|x_{n+1}|} < \varepsilon$$

- Residual

$$|f(x_{n+1})| < \varepsilon$$

assume $f(x) = 0$ for some x

In practice, use a combination for robustness

Stopping Criteria

Always include a **maximum iteration limit** to prevent infinite loops

```
1 iter = 0
2 max_iter = 100
3 while error > tol and iter < max_iter:
4     # Newton-Raphson iteration formula
5     iter += 1
6 if iter >= max_iter:
7     print('Warning: Maximum iterations reached')
```

Intuitive Understanding of Newton-Raphson Method

Given the **starting point** x_n for finding the root of the equation $f(x) = 0$

- Locate the point on the curve at this guess: $(x_n, f(x_n))$
- Draw the **tangent line** to the curve at this point
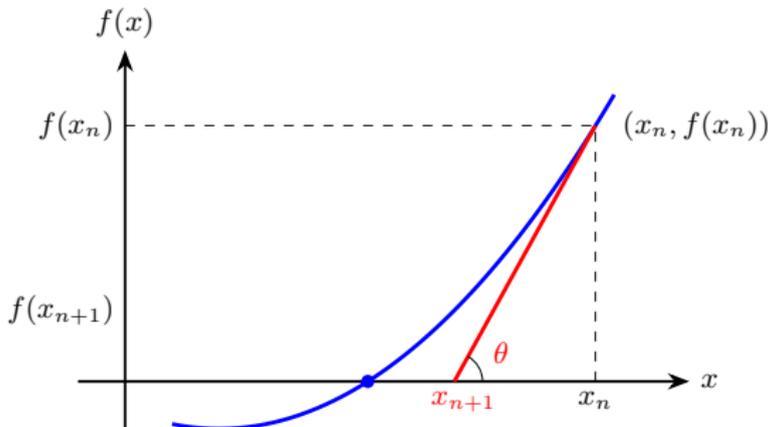- Observe where this **tangent line crosses the x-axis**: $(x_{n+1}, 0)$
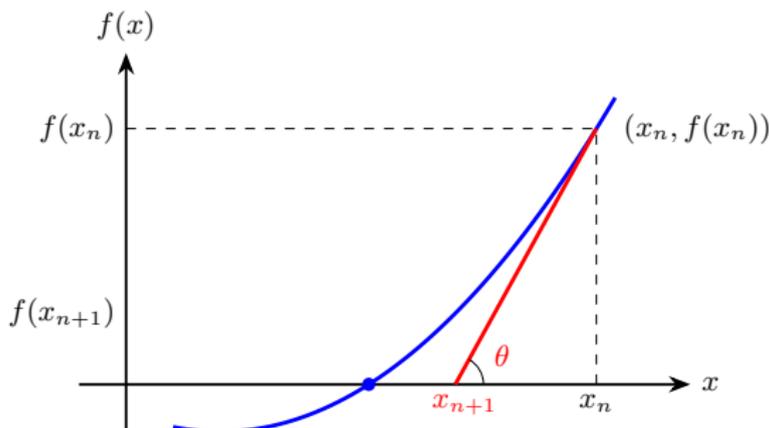
Intuitive Understanding of Newton-Raphson Method

Slope $f'(x_n)$:

- Derivative of function $f(x)$ evaluated at x_n
- Definition of the slope of a line:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{0 - f(x_n)}{x_{n+1} - x_n}$$

Intuitive Understanding of Newton-Raphson Method

We start with the equation:

$$f'(x_n) = \frac{-f(x_n)}{x_{n+1} - x_n}$$

1. Multiply both sides by $(x_{n+1} - x_n)$:

$$f'(x_n) \cdot (x_{n+1} - x_n) = -f(x_n)$$

2. Divide both sides by $f'(x_n)$:

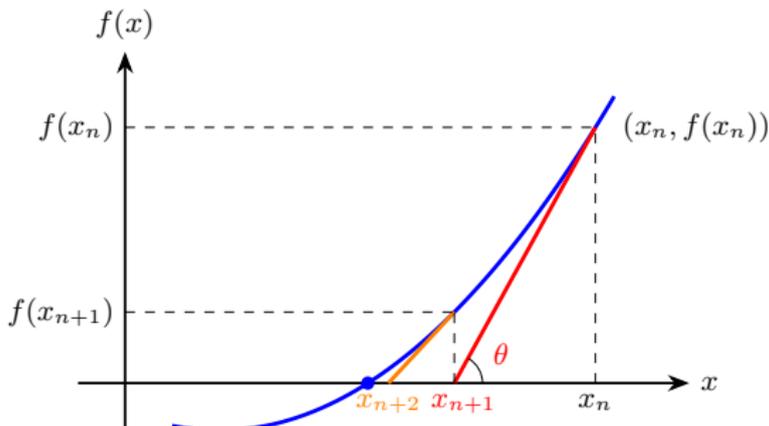$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$

3. Add x_n to both sides to isolate x_{n+1}:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Intuitive Understanding of Newton-Raphson Method

How about slope $f'(x_{n+1})$?

$$f'(x_{n+1}) = \frac{0 - f(x_{n+1})}{x_{n+2} - x_{n+1}}$$

Newton-Raphson Method
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Application
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Convergence
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Drawbacks
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Intuitive Understanding of Newton-Raphson Method

How about slope $f'(x_{n+1})$?

$$f'(x_{n+1}) = \frac{0 - f(x_{n+1})}{x_{n+2} - x_{n+1}}$$

1. Multiply both sides by $(x_{n+2} - x_{n+1})$:

$$f'(x_{n+1}) \cdot (x_{n+2} - x_{n+1}) = -f(x_{n+1})$$

2. Divide both sides by $f'(x_{n+1})$:

$$x_{n+2} - x_{n+1} = -\frac{f(x_{n+1})}{f'(x_{n+1})}$$

3. Add x_{n+1} to both sides to isolate x_{n+2}:

$$x_{n+2} = x_{n+1} - \frac{f(x_{n+1})}{f'(x_{n+1})}$$

Solving $x^2 - 2 = 0$

Given initial point $x_0 = 1.5$, solve $x^2 - 2 = 0$

- Function $f(x) = x^2 - 2$
- Derivative $f'(x) = 2x$
- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$$

Solving $x^2 - 2 = 0$

Given initial point $x_0 = 1.5$, solve $x^2 - 2 = 0$

- Function $f(x) = x^2 - 2$
- Derivative $f'(x) = 2x$
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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$$

- Newton-Raphson method:
    ○ First iteration: $x_1 = \dfrac{x_0}{2} + \dfrac{1}{x_0} = 1.416666667$
    ○ Second iteration: $x_2 = \dfrac{x_1}{2} + \dfrac{1}{x_1} = 1.414215686$
    ○ Third iteration: $x_3 = \dfrac{x_2}{2} + \dfrac{1}{x_2} = 1.414213562$
    ○ Fourth iteration: $x_4 = \dfrac{x_3}{2} + \dfrac{1}{x_3} = 1.414213562$

Converges to $\sqrt{2} = 1.414213562$ in just **3 iterations**!

Solving $x^2 - 2 = 0$

```python
1  def newton_raphson_sqrt2(x0, tol=1e-9, max_iter=100):
2      n = 0
3      xn = x0
4      while n < max_iter:
5          # Calculate f(xn) and f'(xn)
6          fx = xn**2 - 2
7          f_prime = 2 * xn
8          # Newton-Raphson Formula
9          x_next = xn - (fx / f_prime)
10          # Calculate Relative Error
11          error = abs(x_next - xn) / abs(x_next)
12          # Check Convergence
13          if error < tol:
14              return x_next
15          xn = x_next
16          n += 1
17      return xn
18
19  # Run with initial point x0 = 1.5
20  root = newton_raphson_sqrt2(1.5)
```

Quick Summary

Monday's Class:

- Newton-Raphson iteration formula: Derivation from Taylor series
- Intuitive understanding of Newton-Raphson method
- Solve the nonlinear equation $x^2 - 2 = 0$

Quizzes Now!

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  "`Class Participation Quiz 14`"

  Time slot: **2:30PM – 3:00PM**

  on Canvas.

Solving $x^3 - 20 = 0$

How to solve $f(x) = x^3 - 20 = 0$ by Newton-Raphson method?

- First-order Taylor series approximation:

$$f(x_{n+1}) \approx f(x_n) + f'(x_n)(\underbrace{x_{n+1}}_{\text{unknown?}} - x_n)$$

- Let both sides be zero while x_{n+1} is unknown:

Solving $x^3 - 20 = 0$

How to solve $f(x) = x^3 - 20 = 0$ by Newton-Raphson method?

- First-order Taylor series approximation:

$$f(x_{n+1}) \approx f(x_n) + f'(x_n)(\underbrace{x_{n+1}}_{\text{unknown?}} -x_n)$$

- Let both sides be zero while x_{n+1} is unknown:

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0$$
$$f'(x_n)(x_{n+1} - x_n) = -f(x_n)$$
$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which is the **Newton-Raphson iteration formula** at iteration n

- Initial point $x_0 = 3$

Newton-Raphson Method
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Application
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Convergence
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Drawbacks
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Solving $x^3 - 20 = 0$

How to solve $f(x) = x^3 - 20 = 0$ by Newton-Raphson method?

- Derivative $f'(x) = 3x^2$
- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 20}{3x_n^2}$$

- Initial point $x_0 = 3$
- **First iteration**:

$$x_1 = x_0 - \frac{x_0^3 - 20}{3x_0^2}$$
$$= 3 - \frac{3^3 - 20}{3 \times 3^2}$$
$$= 3 - \frac{27 - 20}{27} = 3 - \frac{7}{27} = 2.74074$$

Solving $x^3 - 20 = 0$

How to solve $f(x) = x^3 - 20 = 0$ by Newton-Raphson method?

- Derivative $f'(x) = 3x^2$
- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 20}{3x_n^2}$$

- Initial point $x_0 = 3$
- **First iteration**:

$$x_1 = x_0 - \frac{x_0^3 - 20}{3x_0^2}$$
$$= 3 - \frac{3^3 - 20}{3 \times 3^2}$$
$$= 3 - \frac{27 - 20}{27} = 3 - \frac{7}{27} = 2.74074$$

- **Relative error**:

$$\varepsilon_1 = \frac{|x_1 - x_0|}{|x_1|} = \frac{|2.74074 - 3|}{|2.74074|} = 9.46\%$$

Solving $x^3 - 20 = 0$

How to solve $f(x) = x^3 - 20 = 0$ by Newton-Raphson method?

- Derivative $f'(x) = 3x^2$
- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 20}{3x_n^2}$$

- Starting point $x_1 = 2.74074$
- **Second iteration**:

$$x_2 = x_1 - \frac{x_1^3 - 20}{3x_1^2}$$

$$= 2.74074 - \frac{2.74074^3 - 20}{3 \times 2.74074^2}$$

$$= 2.74074 - 0.02607 = 2.71467$$

- **Relative error**:

$$\varepsilon_2 = \frac{|x_2 - x_1|}{|x_2|} = \frac{|2.71467 - 2.74074|}{|2.71467|} = 0.96\%$$

Solving $x^3 - 20 = 0$

How to solve $f(x) = x^3 - 20 = 0$ by Newton-Raphson method?

- Derivative $f'(x) = 3x^2$
- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 20}{3x_n^2}$$

- Starting point $x_2 = 2.71467$
- **Third iteration**:

$$\begin{aligned} x_3 &= x_2 - \frac{x_2^3 - 20}{3x_2^2} \\ &= 2.71467 - \frac{2.71467^3 - 20}{3 \times 2.71467^2} \\ &= 2.71467 - 0.00025 = 2.71442 \end{aligned}$$

- **Relative error**:

$$\varepsilon_3 = \frac{|x_3 - x_2|}{|x_3|} = \frac{|2.71442 - 2.71467|}{|2.71442|} = 0.009\%$$

Solving $x^3 - 20 = 0$

```python
1  def newton_raphson_x_cubic(x0, tol=1e-9, max_iter=100):
2      n = 0
3      xn = x0
4      while n < max_iter:
5          # Calculate f(xn) and f'(xn)
6          fx = xn**3 - 20
7          f_prime = 3 * xn**2
8          # Newton-Raphson Formula
9          x_next = xn - (fx / f_prime)
10         # Calculate Relative Error
11         error = abs(x_next - xn) / abs(x_next)
12         print(x_next)
13         print(error * 100)
14         print()
15         # Check Convergence
16         if error < tol:
17             return x_next
18         xn = x_next
19         n += 1
20     return xn
21
22  x = newton_raphson_x_cubic(3)
```

Quadratic Convergence

- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Let the root of $f(x) = 0$ be $x^\star$ (actual root)

- Add $-x^\star$ to both sides:

$$\underbrace{x_{n+1} - x^\star}_{\text{residual } \varepsilon_{n+1}} = \underbrace{x_n - x^\star}_{\text{residual } \varepsilon_n} - \frac{f(x_n)}{f'(x_n)}$$

or

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(x_n)}{f'(x_n)}$$

with residual

$$\varepsilon_n = x_n - x^\star$$

Quadratic Convergence (Optional)

- Expand $f(b)$ around a using a second-order Taylor polynomial:

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(\beta)}{2!}(b - a)^2$$

  for some β between a and b.

- Expand $f(x^\star)$ around starting point x_n (defining residual $\varepsilon_n = x_n - x^\star$):

$$f(x^\star) = f(x_n) + f'(x_n) \underbrace{(x^\star - x_n)}_{= -\varepsilon_n} + \frac{f''(\beta)}{2!} \underbrace{(x^\star - x_n)^2}_{= \varepsilon_n^2}$$

  where β is some value between $x^\star$ and x_n.

Newton-Raphson Method
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Application
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Convergence
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Drawbacks
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Quadratic Convergence (Optional)

- Expand $f(b)$ around a using a second-order Taylor polynomial:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(\beta)}{2!}(b-a)^2$$

for some β between a and b.

- Expand $f(x^\star)$ around starting point x_n (defining residual $\varepsilon_n = x_n - x^\star$):

$$f(x^\star) = f(x_n) + f'(x_n)\underbrace{(x^\star - x_n)}_{=\,-\varepsilon_n} + \frac{f''(\beta)}{2!}\underbrace{(x^\star - x_n)^2}_{=\,\varepsilon_n^2}$$

where β is some value between $x^\star$ and x_n.

- Since $f(x^\star) = 0$ and $\varepsilon_n = x_n - x^\star$, we have

$$0 = f(x_n) - f'(x_n)\varepsilon_n + \frac{f''(\beta)}{2}\varepsilon_n^2$$

$$\Rightarrow \quad 0 = \frac{f(x_n)}{f'(x_n)} - \varepsilon_n + \frac{f''(\beta)}{2f'(x_n)}\varepsilon_n^2 \quad \Rightarrow \quad \varepsilon_n - \frac{f(x_n)}{f'(x_n)} = \frac{f''(\beta)}{2f'(x_n)}\varepsilon_n^2$$

Quadratic Convergence (Optional)

- Recall that

$$\varepsilon_n = x_n - x^\star$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(x_n)}{f'(x_n)}$$

we have

$$\varepsilon_n - \frac{f(x_n)}{f'(x_n)} = \frac{f''(\beta)}{2f'(x_n)}\varepsilon_n^2 \quad \Rightarrow \quad \varepsilon_{n+1} = \frac{f''(\beta)}{2f'(x_n)}\varepsilon_n^2$$

Quadratic Convergence

- Quadratic convergence established!

$$|\varepsilon_{n+1}| = \left| \frac{f''(\beta)}{2f'(x_n)} \right| \cdot \varepsilon_n^2$$

As x_n approaches $x^\star$, the term $\left| \frac{f''(\beta)}{2f'(x_n)} \right|$ approaches a constant C

- Asymptotic error constant:

$$C = \left| \frac{f''(x^\star)}{2f'(x^\star)} \right|$$

  - C determines the speed of convergence
  - Small C implies faster convergence
  - If $f''(x^\star) = 0$, convergence can be even faster

- The order of convergence is 2:

$$|\varepsilon_{n+1}| \approx C \cdot \varepsilon_n^2$$

Quadratic Convergence

- Asymptotic error constant:

$$C = \left| \frac{f''(x^\star)}{f'(x^\star)} \right|$$

  Or

$$|\varepsilon_{n+1}| \approx C \cdot \varepsilon_n^2 \quad \Rightarrow \quad \frac{|\varepsilon_{n+1}|}{|\varepsilon_n|^2} = \frac{|x_{n+1} - x^\star|}{|x_n - x^\star|^2} \approx C$$
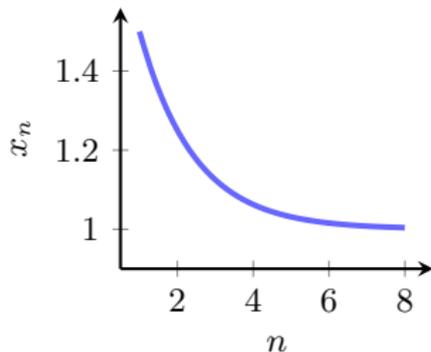
- Example:

$$f(x) = x^2 - 2$$

Quadratic Convergence

- Asymptotic error constant:

$$C = \left| \frac{f''(x^\star)}{f'(x^\star)} \right|$$

  Or

$$|\varepsilon_{n+1}| \approx C \cdot \varepsilon_n^2 \quad \Rightarrow \quad \frac{|\varepsilon_{n+1}|}{|\varepsilon_n|^2} = \frac{|x_{n+1} - x^\star|}{|x_n - x^\star|^2} \approx C$$

- Example:

$$f(x) = x^2 - 2$$

  ○ Find the root $f(x) = 0 \quad \Rightarrow \quad x^\star = \sqrt{2}$
  ○ Since $f'(x) = 2x$, we have $f'(x^\star) = 2\sqrt{2}$
  ○ Since $f''(x) = 2$, we have $f''(x^\star) = 2$
  ○ As a result, we have

$$C = \left| \frac{2}{4\sqrt{2}} \right| \approx 0.3535$$

Rate of Convergence

Linear convergence

$$\frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} \le r,\ r \in (0,1)$$

Sequence $x_n = 1 + \left(\frac{1}{2}\right)^n$ converges linearly to $x_\infty = 1$ because $r = \frac{1}{2}$
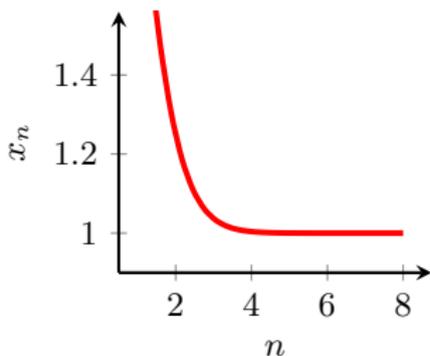
Rate of Convergence
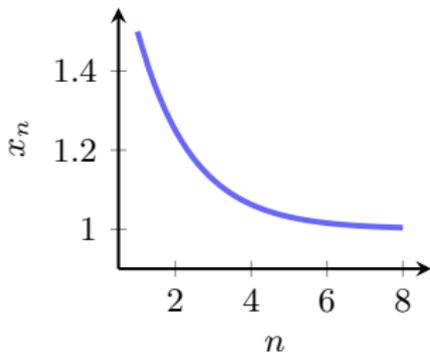
Linear convergence

$$\frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} \leq r,\ r \in (0,1)$$

Sequence $x_n = 1 + \left(\frac{1}{2}\right)^n$
converges linearly to
$x_\infty = 1$ because $r = \frac{1}{2}$

Superlinear convergence

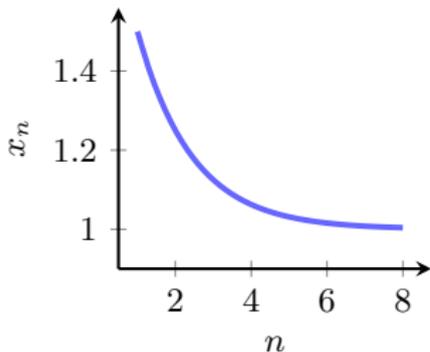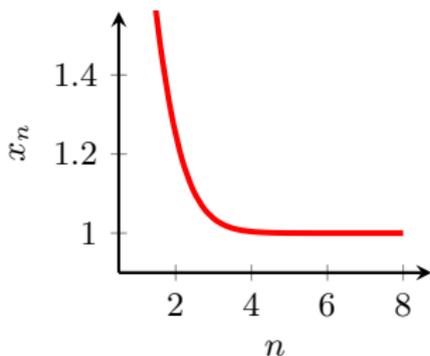$$\lim_{n \to \infty} \frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} = 0$$

Sequence $x_n = 1 + \left(\frac{1}{n}\right)^n$
converges superlinearly to
$x_\infty = 1$

Rate of Convergence

Linear convergence

$$\frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} \le r,\, r \in (0,1)$$

Sequence $x_n = 1 + \left(\frac{1}{2}\right)^n$
converges linearly to
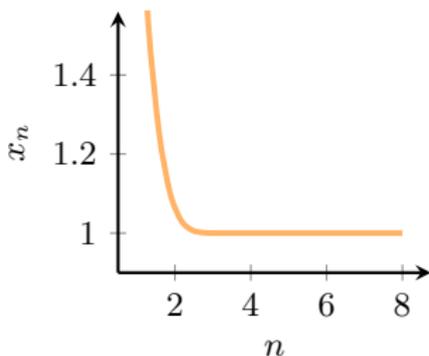$x_\infty = 1$ because $r = \frac{1}{2}$

Superlinear convergence

$$\lim_{n \to \infty} \frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} = 0$$

Sequence $x_n = 1 + \left(\frac{1}{n}\right)^n$
converges superlinearly to
$x_\infty = 1$

Quadratic convergence
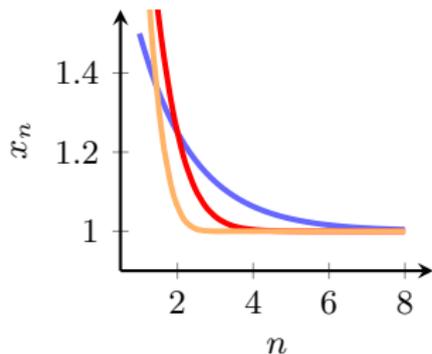
$$\frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|^2} \le M$$

Sequence $x_n = 1 + \left(\frac{1}{n}\right)^{2^n}$
converges quadratically to
$x_\infty = 1$ because $M = 1$
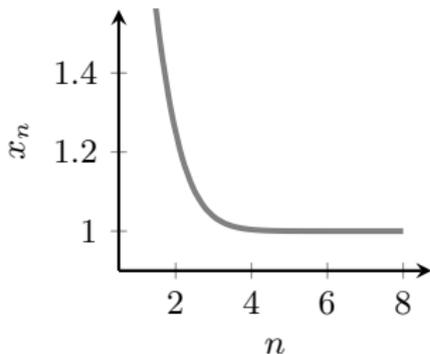
Rate of Convergence

Linear convergence

$$\frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} \leq r,\ r \in (0,1)$$

Sequence $x_n = 1 + \left(\frac{1}{2}\right)^n$
converges linearly to
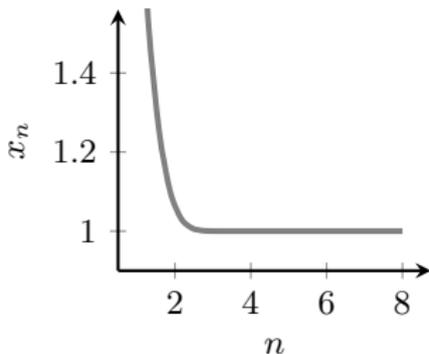$x_\infty = 1$ because $r = \frac{1}{2}$

Superlinear convergence

$$\lim_{n \to \infty} \frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} = 0$$

Sequence $x_n = 1 + \left(\frac{1}{n}\right)^n$
converges superlinearly to
$x_\infty = 1$

Quadratic convergence

$$\frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|^2} \leq M$$

Sequence $x_n = 1 + \left(\frac{1}{n}\right)^{2^n}$
converges quadratically to
$x_\infty = 1$ because $M = 1$

Quick Summary

Wednesday's Class:

- Solve $f(x) = x^3 - 20 = 0$ by Newton-Raphson method
- Describe the quadratic convergence of Newton-Raphson method
  - Newton-Raphson iteration formula for residual ε_n

  $$\varepsilon_{n+1} = \varepsilon_n - \frac{f(x_n)}{f'(x_n)}$$

  - Second-order Taylor polynomial

  $$\underbrace{f(x^\star)}_{=0} = f(x_n) + f'(x_n) \underbrace{(x^\star - x_n)}_{=-\varepsilon_n} + \frac{f''(\beta)}{2} \underbrace{(x^\star - x_n)^2}_{=\varepsilon_n^2}$$

- Comparison among linear, superlinear, and quadratic convergence

Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

    "Class Participation Quiz 15"

    Time slot: **2:30PM – 3:00PM**

  on Canvas.

Rate of Convergence

A sequence $\{x_n\}$ converges to $x^\star$ if:

$$\lim_{n \to \infty} |x_n - x^\star| = 0$$

- **Rate of convergence** describes **how fast** the sequence approaches $x^\star$

Rate of Convergence

- Let

$$\varepsilon_n = x_n - x^\star$$

be the residual at iteration n

Iteration (n)	Linear $(\varepsilon_{n+1} = 0.5\varepsilon_n)$	Quadratic $(\varepsilon_{n+1} = \varepsilon_n^2)$
0	0.1	0.1
1	0.05	0.01
2	0.025	0.0001
3	0.0125	10^{-8}
4	0.00625	10^{-16}

Quadratic convergence: **Digits double each iteration**!

Newton-Raphson Method
○○○○○○○○○○○○○○○○○○○
Application
○○○○○
Convergence
○○○○○○○○○○
Drawbacks
●○○○○○○○○○○○○○○

Divergence

❶ **Divergence at inflection points**:

- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \underbrace{\frac{f(x_n)}{f'(x_n)}}_{\text{correction term}}$$

- **Critical requirement**: $f'(x_n) \neq 0$
- When $f'(x_n) \approx 0$?
  - ○ Small denominator → very correction term
  - ○ Next point x_{n+1} may be far from current point
  - ○ May lead to **divergence**

Divergence

Problem statement: Find the root of $f(x) = (x-1)^3 + 0.512 = 0$

- **Root**: Solve $(x-1)^3 = -0.512$

$$x - 1 = \sqrt[3]{0.512} = -0.8 \quad \Rightarrow \quad x = 0.2$$

- First-order derivative: $f'(x) = 3(x-1)^2$

Newton-Raphson Method
○○○○○○○○○○○○○○○○○○○

Application
○○○○○

Convergence
○○○○○○○○○○

Drawbacks
○●○○○○○○○○○○○○○○

Divergence

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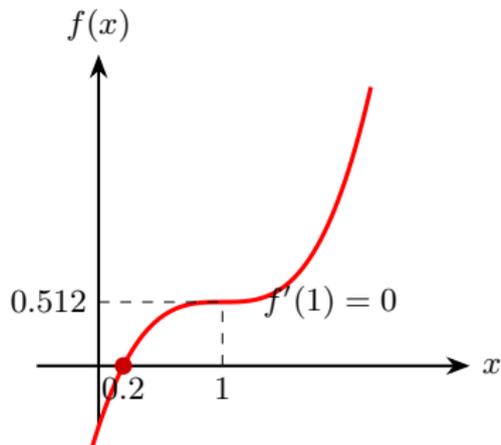
- First-order derivative: $f'(x) = 3(x-1)^2$
- **Infection point**: At $x = 1$, we have $f'(1) = 0$

Divergence

Problem statement: Find the root of $f(x) = (x-1)^3 + 0.512 = 0$

- Starting with $x_0 = 5$

- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- First 5 iterations:

Iteration (n)	x_n
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589

Divergence

Iteration (n)	x_n
4	1.6000
5	0.92589
6	-30.119
7	-19.746
8	-12.831
9	-8.2217
10	-5.1498
11	-3.1044
12	-1.7464
13	-0.85356
14	-0.28538
15	0.039784
16	0.17475
17	0.19924
18	0.2

Observation: Progressively moving toward the inflection point at $x = 1$

Divergence

Divergence near an inflection point ($x = 1$)

- Starting with $x_0 = 5$
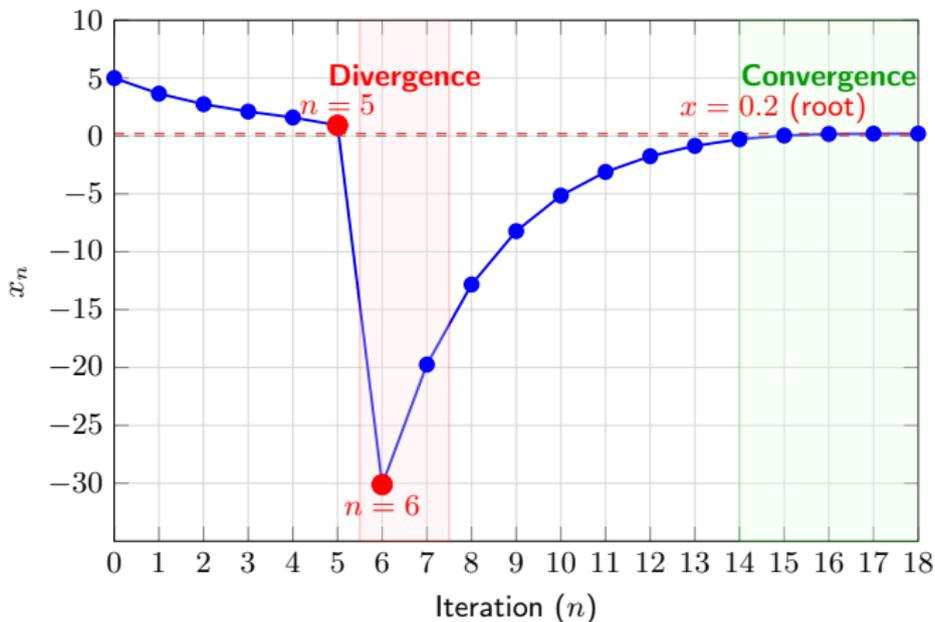
Divergence

```python
1  def func(x0, tol=1e-9, max_iter=100):
2      n = 0
3      xn = x0
4      while n < max_iter:
5          fx = (xn - 1) ** 3 + 0.512
6          f_prime = 3 * (xn - 1) ** 2
7          x_next = xn - (fx / f_prime)
8          error = abs(x_next - xn) / abs(x_next)
9          print(n)
10         print(x_next)
11         print(error * 100)
12         print()
13         # Check Convergence
14         if error < tol:
15             return x_next
16         xn = x_next
17         n += 1
18     return xn
```

Division

❷ **Division by zero and near-zero derivatives**:

- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- If $f'(x_n) = 0$, cannot compute next iterate
- Even when division by zero is avoided, near-zero derivatives can cause serious convergence problems!
- **Example**: Find the root of $f(x) = (x-1)^3 + 0.512 = 0$, starting with $x_0 = 1.001$

```
1 x = func (1.001)
```

Oscillation

❸ **Oscillations near local extrema**:

- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- **Problem**: When starting near a local extrema, the method may:
    - Oscillate between two points (an infinite cycle)
    - Never converge to a root

Oscillation

Problem statement: Find the root of $f(x) = x^3 - 5x = 0$

- **Roots**: $x = 0, \pm\sqrt{5} \approx 2.236$
- First-order derivative: $f'(x) = 3x^2 - 5$
- **Infection point**: At $x = \pm\sqrt{5/3} \approx \pm 1.291$

Oscillation

Problem statement: Find the root of $f(x) = x^3 - 5x = 0$

- Starting with $x_0 = 1$:

  - $f(x_0) = -4$
  - $f'(x_0) = -2$

  $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - 2 = -1$$

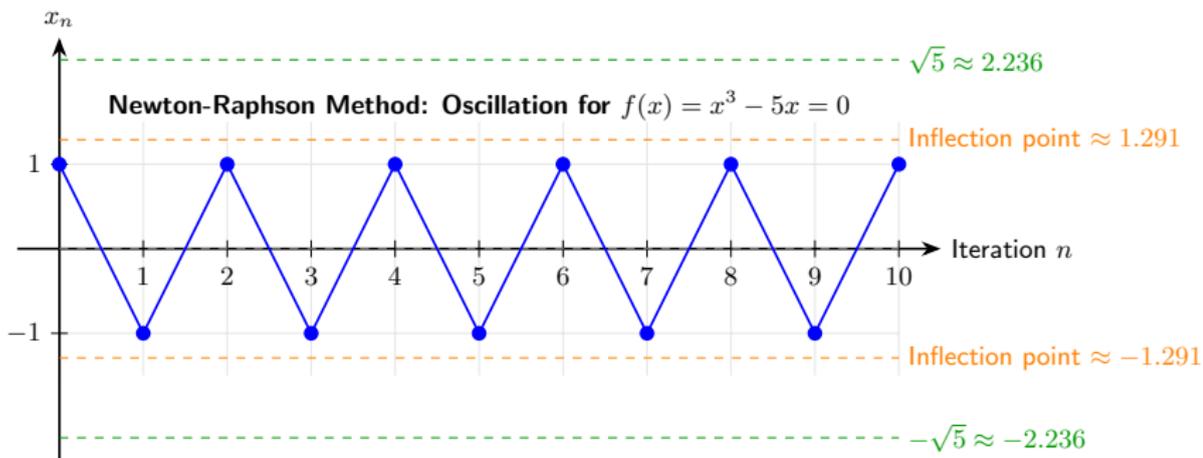- Now at $x_1 = -1$:

  - $f(x_1) = 4$
  - $f'(x_1) = -2$

  $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1 + 2 = 1$$

  The method alternates between $x = 1$ and $x = -1$ indefinitely

Remarks:

- Oscillation occurs when starting near local extrema
- The method may never converge to any root

Oscillation

Initialization

❹ **Sensitive to initial points**:

- Finding the roots of $f(x) = x^3 - 2x^2 - 11x + 12 = 0$
- 5 initial points x_0 with marginal difference

Initial point x_0	Result $x^\star$
2.352875270	converges to 4
2.352841720	

Initialization

❹ **Sensitive to initial points**:

- Finding the roots of $f(x) = x^3 - 2x^2 - 11x + 12 = 0$
- 5 initial points x_0 with marginal difference

Initial point x_0	Result $x^\star$
2.352875270	converges to 4
2.352841720	converges to -3
2.352837350	converges to 4
2.352836327	converges to -3
2.352836323	converges to 1

Linear Convergence

❺ **Converge only in linear time**:
- Example: Finding the root of $f(x) = x^2 = 0$
  - First-order derivative: $f'(x) = 2x$
  - Second-order derivative: $f''(x) = 2$
- Newton-Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2}{2x_n} = \frac{x_n}{2}$$

Linear Convergence

❺ **Converge only in linear time**:

- Example: Finding the root of $f(x) = x^2 = 0$
    - First-order derivative: $f'(x) = 2x$
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✗ Quadratic convergence:

$$\lim_{n \to \infty} \frac{|x_{n+1} - x^\star|}{|x_n - x^\star|^2} = \lim_{n \to \infty} \frac{x_n/2}{x_n^2} = \lim_{n \to \infty} \frac{1}{2x_n} = \infty \quad \text{(unbounded)}$$

Linear Convergence

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- Example: Finding the root of $f(x) = x^2 = 0$
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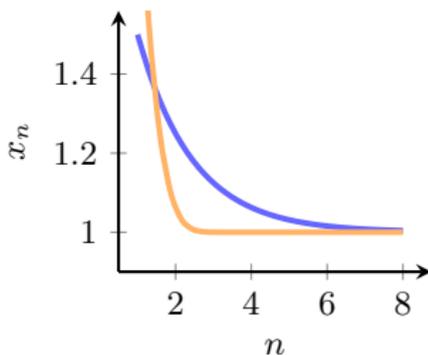
✓ Linear convergence:

$$\lim_{n \to \infty} \frac{|x_{n+1} - x^\star|}{|x_n - x^\star|} = \lim_{n \to \infty} \frac{x_n/2}{x_n} = \frac{1}{2}$$

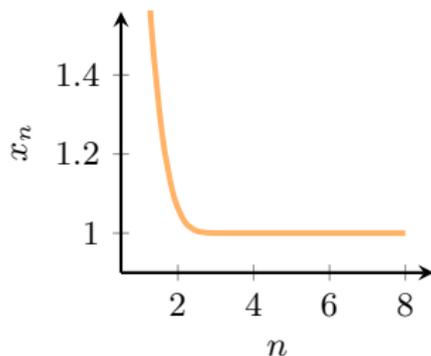## Rate of Convergence

Linear convergence

$$\frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} \le r,\, r \in (0, 1)$$

Sequence $x_n = 1 + \left(\frac{1}{2}\right)^n$
converges linearly to
$x_\infty = 1$ because $r = \frac{1}{2}$

Quadratic convergence

$$\frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|^2} \le M$$

Sequence $x_n = 1 + \left(\frac{1}{n}\right)^{2^n}$
converges quadratically to
$x_\infty = 1$ because $M = 1$

Quick Summary

Friday's Class:

- Divergence at inflection points
- Division by zero and near-zero derivatives
- Oscillations near local extrema
- Sensitive to initial points
- Linear convergence (special case such as $x^2 = 0$)

Newton-Raphson method is powerful but also has drawbacks, please monitor its behavior.