Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 12: Optimization techniques: Part I

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Linear Programming

- **Concept**: Linear programming or linear optimization
- A special case of mathematical programming
- **Definition**: Linear programming is a technique for the optimization of a **linear objective function**, subject to (s.t.) **linear equality and linear inequality constraints**.
- **Description**: Find the **best outcome** (such as maximum profit or lowest cost) in a model where requirements are represented by **linear** relationships.

Mathematical Programming

- **A general formula**:

$$\min_x \quad f(x) \qquad \text{or} \qquad \max_x \quad f(x)$$
$$\text{s.t.} \quad x \in \mathcal{C} \qquad\qquad\qquad \text{s.t.} \quad x \in \mathcal{C}$$

- **Main components**:
  - Decision variable x
  - Objective function $f(x)$
  - Linear constraint $x \in \mathcal{C}$
  - Feasible set $\mathcal{C}$, e.g., $\mathcal{C} = \{x \mid x \geq 0\}$
- **Format**: Minimize ($\min$) vs. Maximize ($\max$)

Linear Programming

- **Example**:

$$\begin{aligned}\max_{x_1, x_2} \quad & c_1 x_1 + c_2 x_2 && \text{(linear objective)}\\ \text{s.t.} \quad & a_{11} x_1 + a_{12} x_2 \leq b_1 && \text{(linear inequality)}\\ & a_{21} x_1 + a_{22} x_2 \leq b_2 && \text{(linear inequality)}\\ & x_1, x_2 \geq 0 && \text{(non-negativity)}\end{aligned}$$

or

$$\begin{aligned}\max_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x} && \text{(linear objective)}\\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} && \text{(linear inequality)}\\ & \boldsymbol{x} \geq 0 && \text{(non-negativity)}\end{aligned}$$

for any matrix $\boldsymbol{A}$ and vectors $\boldsymbol{b}, \boldsymbol{c}$.

Geometry of Linear Programming

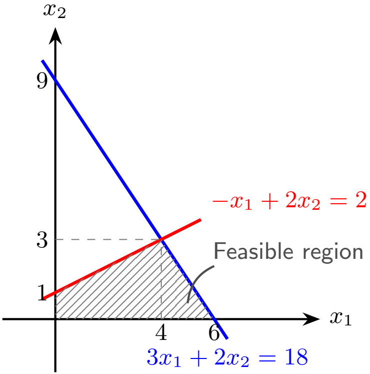
Linear programming:

$$\begin{aligned} \max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \leq 18 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Geometry of Linear Programming

Linear programming:

$$
\begin{aligned}
\max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & x_1 + x_2 \\
\text{s.t.} \quad & 3x_1 + 2x_2 \leq 18 \\
& -x_1 + 2x_2 \leq 2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



Linear constraints define the **feasible region**, which is a convex polytope.

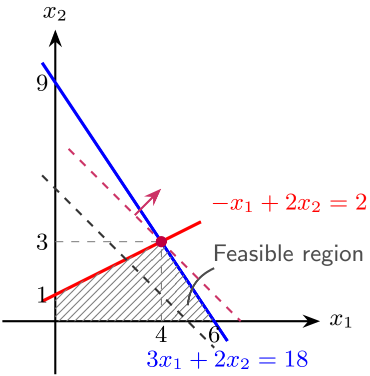
Geometry of Linear Programming

Linear programming:

$$\max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad x_1 + x_2$$
$$\text{s.t.} \quad 3x_1 + 2x_2 \leq 18$$
$$-x_1 + 2x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Optimal solution:

$$x_1^\star = 4 \quad x_2^\star = 3$$



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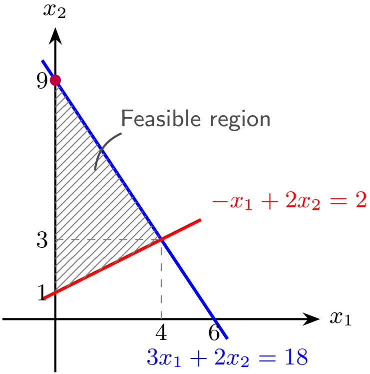
$$-x_1 + 2x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Optimal solution:

$$x_1^\star = 0 \quad x_2^\star = 9$$



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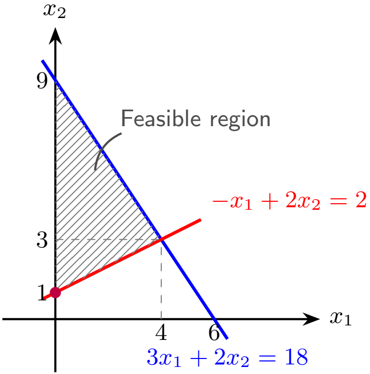
Geometry of Linear Programming

Linear programming:

$$\begin{aligned} \min_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \leq 18 \\ & -x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Optimal solution:

$$x_1^\star = 0 \quad x_2^\star = 1$$



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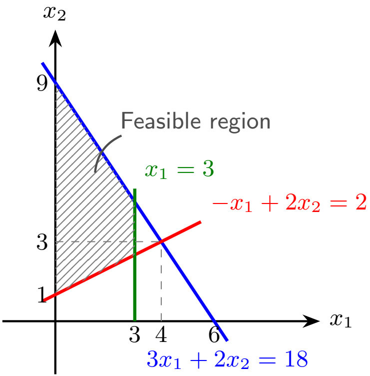
Geometry of Linear Programming

Linear programming:

$$\begin{aligned} \max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & 4x_1 + x_2 \\ \text{s.t.} \quad & x_1 \leq 3 \\ & 3x_1 + 2x_2 \leq 18 \\ & -x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Optimal solution:

$$x_1^\star = 3 \quad x_2^\star = 4.5$$



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Geometry of Linear Programming

Linear programming:

$$\begin{aligned} \max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & 4x_1 + x_2 \\ \text{s.t.} \quad & x_1 \leq 3 \\ & 3x_1 + 2x_2 \leq 18 \\ & -x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

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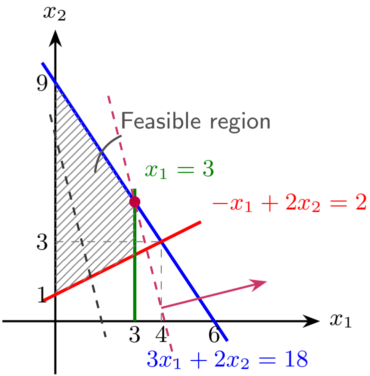
$$x_1^\star = 3 \quad x_2^\star = 4.5$$



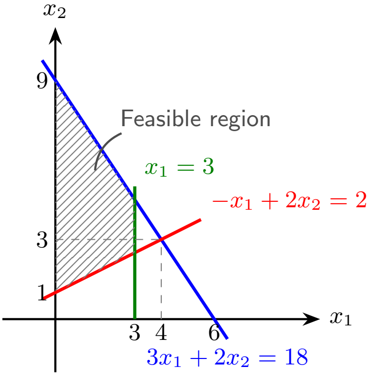
Linear constraints define the **feasible region**, which is a convex polytope.

Geometry of Linear Programming

Linear programming:

$$\begin{aligned} \min_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & -x_1 + x_2 \\ \text{s.t.} \quad & x_1 \leq 3 \\ & 3x_1 + 2x_2 \leq 18 \\ & -x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Optimal solution?



Linear constraints define the **feasible region**, which is a convex polytope.

Python Programming

```python
import cvxpy as cp

# Define the decision variables
x = cp.Variable(2, nonneg=True)
# Define the objective function
objective = cp.Minimize(-x[0] + x[1])

# Define the constraints
constraints = [
    x[0] <= 3,                          # x1 <= 3
    3*x[0] + 2*x[1] <= 18,              # 3x1 + 2x2 <= 18
    -x[0] + 2*x[1] >= 2                 # -x1 + 2x2 >= 2
]

# Formulate and solve the problem
prob = cp.Problem(objective, constraints)
result = prob.solve()
# Output the results
print(f'Status: {prob.status}')
print(f'Optimal Value (max objective): {prob.value:.4f}')
print(f'Optimal x1: {x.value[0]:.4f}')
print(f'Optimal x2: {x.value[1]:.4f}')
```

ℓ_1-Norm Optimization

Given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$, find the best $\boldsymbol{x} \in \mathbb{R}^n$ that minimizes

$$\boldsymbol{\varepsilon} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}$$

- The sum of relative residuals:

$$\underbrace{\|\boldsymbol{\varepsilon}\|_1}_{\ell_1\text{-norm}} = \sum_{i=1}^{n} |\varepsilon_i|$$

- ℓ_1-norm optimization:

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & \sum_{i=1}^{n} |\varepsilon_i| \\ \text{s.t.} \quad & \boldsymbol{\varepsilon} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \end{aligned}$$

ℓ_1-Norm Optimization

- Linear programming with bound variables $c_i, i = 1, 2, \ldots, n$:

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n, \boldsymbol{c}\in\mathbb{R}^n} \quad & \sum_{i=1}^{n} c_i \\ \text{s.t.} \quad & \boldsymbol{\varepsilon} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \\ & -c_i \leq \varepsilon_i \leq c_i \quad i = 1, 2, \ldots, n \\ & c_i \geq 0 \quad i = 1, 2, \ldots, n \end{aligned}$$

- Toy example:

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 18 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \text{Exact solution:} \quad \boldsymbol{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

ℓ_1-Norm Optimization

```python
import cvxpy as cp
import numpy as np

n = 2
A = np.array([[3, 2], [-1, 2]])
b = np.array([18, 2])

# Define decision variables
x = cp.Variable(n)
c = cp.Variable(n, nonneg=True)
# Minimize the sum of the bound variables c_i
objective = cp.Minimize(cp.sum(c))
# Define constraints
constraints = [-c <= b - A @ x, b - A @ x <= c]
# Formulate and solve the problem
prob = cp.Problem(objective, constraints)
prob.solve()
# Output results
print(f'Status: {prob.status}')
print(f'Optimal objective value (sum of c): {prob.value}')
print(f'Optimal x:\n{x.value}')
```

ℓ_∞-Norm Optimization

Given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$, find the best $\boldsymbol{x} \in \mathbb{R}^n$ that minimizes

$$\boldsymbol{\varepsilon} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}$$

- The worst-case residual:

$$\underbrace{\|\boldsymbol{\varepsilon}\|_\infty}_{\ell_\infty\text{-norm}} = \max\{|\varepsilon_1|, |\varepsilon_2|, \ldots, |\varepsilon_n|\}$$

- ℓ_∞-norm optimization:

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & \max\{|\varepsilon_1|, |\varepsilon_2|, \ldots, |\varepsilon_n|\} \\ \text{s.t.} \quad & \boldsymbol{\varepsilon} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \end{aligned}$$

ℓ_∞-Norm Optimization

- Linear programming with bound variable c (scalar):

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n, c\in\mathbb{R}} \quad & c \\ \text{s.t.} \quad & \boldsymbol{\varepsilon} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \\ & -c \leq \varepsilon_i \leq c \quad i = 1, 2, \ldots, n \\ & c \geq 0 \end{aligned}$$

- Toy example:

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 18 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \text{Exact solution:} \quad \boldsymbol{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

ℓ_∞-Norm Optimization

```python
import cvxpy as cp
import numpy as np

n = 2
A = np.array([[3, 2], [-1, 2]])
b = np.array([18, 2])

# Define decision variables
x = cp.Variable(n)
c = cp.Variable(1, nonneg=True)
# Minimize the bound variable c
objective = cp.Minimize(c)
# Define constraints
constraints = [-c <= b - A @ x, b - A @ x <= c]
# Formulate and solve the problem
prob = cp.Problem(objective, constraints)
prob.solve()
# Output results
print(f'Status: {prob.status}')
print(f'Optimal objective value: {prob.value}')
print(f'Optimal x:\n{x.value}')
```

Quick Summary

Monday's Class:

- Essential formula of optimization
- What is linear programming?
- Geometric interpretation of linear programming
- Applications to ℓ_1- and ℓ_∞-norm optimization

Integer Programming

- **Integer programming** is a mathematical programming in which some or all of the variables are restricted to be integers.

$$\begin{aligned} \text{Integer decision variable:} \quad & x \in \mathbb{Z} \\ \text{Binary decision variable:} \quad & x \in \{0, 1\} \quad \text{(either 0 or 1)} \\ \text{Integer decision vector:} \quad & \boldsymbol{x} \in \mathbb{Z}^n \end{aligned}$$

- The set of all integers is $\mathbb{Z}$: e.g., $-3, -2, -1, 0, 1, 2, 3$

Geometry of Integer Programming

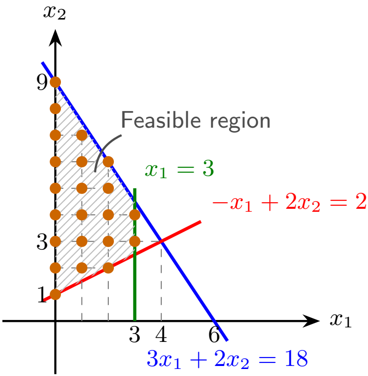
Integer programming:

$$\max_{x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z}} \quad 4x_1 + x_2$$
$$\text{s.t.} \quad x_1 \leq 3$$
$$3x_1 + 2x_2 \leq 18$$
$$-x_1 + 2x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

All feasible solutions (●)

Optimal solution:

$$x_1^\star = 3 \quad x_2^\star = 4$$



Geometry of Integer Programming

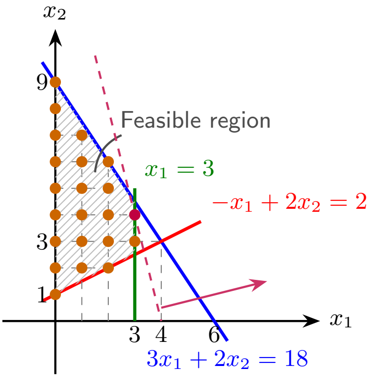
Integer programming:

$$\max_{x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z}} \quad 4x_1 + x_2$$
$$\text{s.t.} \quad x_1 \leq 3$$
$$3x_1 + 2x_2 \leq 18$$
$$-x_1 + 2x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

All feasible solutions (●)

Optimal solution:

$$x_1^\star = 3 \quad x_2^\star = 4$$



Geometry of Integer Programming

Integer programming:

$$\max_{x_1 \in \{0,1\}, x_2 \in \mathbb{Z}} \quad 4x_1 + x_2$$

$$\text{s.t.} \quad 3x_1 + 2x_2 \leq 18$$

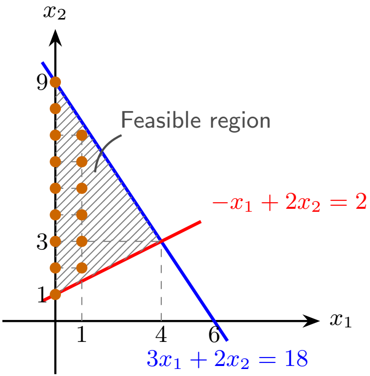
$$-x_1 + 2x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

All feasible solutions (●)

Optimal solution:

$$x_1^\star = 1 \quad x_2^\star = 7$$



Geometry of Integer Programming

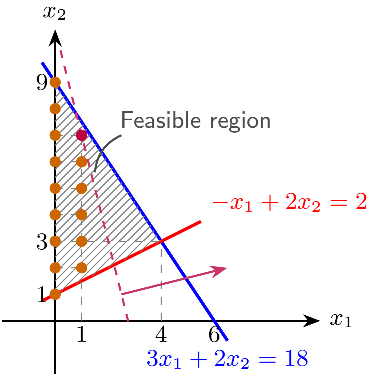
Integer programming:

$$\max_{x_1 \in \{0,1\}, x_2 \in \mathbb{Z}} \quad 4x_1 + x_2$$
$$\text{s.t.} \quad 3x_1 + 2x_2 \leq 18$$
$$-x_1 + 2x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

All feasible solutions (●)

Optimal solution:

$$x_1^\star = 1 \quad x_2^\star = 7$$



Mixed-Integer Programming

- **Mixed-integer programming** is a mathematical programming that solves problems involving a mix of continuous and integer decision variables.
  - Continuous decision variables, e.g., $x \in \mathbb{R}$
  - Integer decision variables, e.g., $x \in \mathbb{Z}$, $x \in \{0, 1\}$

Geometry of Mixed-Integer Programming

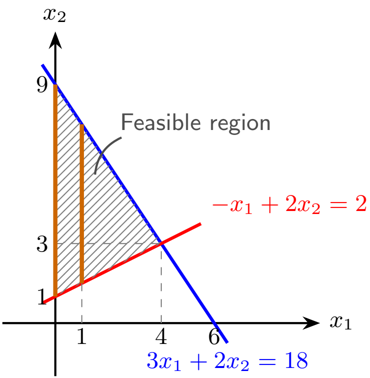
Mixed-integer programming:

$$\max_{x_1 \in \{0,1\}, x_2 \in \mathbb{R}} \quad 4x_1 + x_2$$
$$\text{s.t.} \quad 3x_1 + 2x_2 \leq 18$$
$$-x_1 + 2x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

All feasible solutions (orange line)

Optimal solution:

$$x_1^\star = 1 \quad x_2^\star = 7.5$$



Geometry of Mixed-Integer Programming

Mixed-integer programming:

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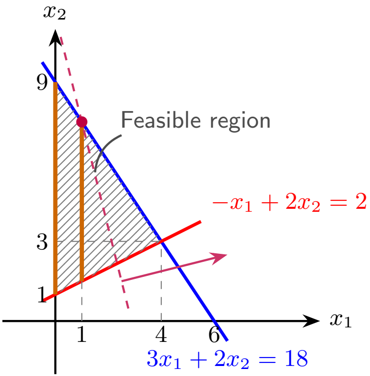
$$\text{s.t.} \quad 3x_1 + 2x_2 \leq 18$$
$$-x_1 + 2x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

All feasible solutions (▬)

Optimal solution:

$$x_1^\star = 1 \quad x_2^\star = 7.5$$



Nonlinear Programming

- **Nonlinear programming** is the process of solving an optimization problem where some of the constraints are not linear equalities or the objective function is not a linear function.

$$\begin{aligned} \min \quad & \text{nonlinear function} \\ \text{s.t.} \quad & \text{nonlinear equality/inequality} \end{aligned}$$

Nonlinear Programming

- **Nonlinear programming** is the process of solving an optimization problem where some of the constraints are not linear equalities or the objective function is not a linear function.

$$\begin{aligned} \min \quad & \text{nonlinear function} \\ \text{s.t.} \quad & \text{nonlinear equality/inequality} \end{aligned}$$

- **Quadratic programming** is the process of solving certain mathematical optimization problems involving quadratic functions.
  - Optimize (minimize/maximize) a multivariate quadratic function subject to linear constraints on the variables
  - A type of nonlinear programming

Geometry of Quadratic Programming

Quadratic programming:

$$\max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad x_1^2 + x_2^2$$

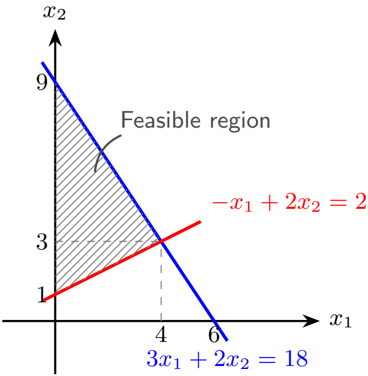
$$\text{s.t.} \quad 3x_1 + 2x_2 \le 18$$

$$-x_1 + 2x_2 \ge 2$$

$$x_1, x_2 \ge 0$$

Optimal solution:

$$x_1^\star = 0 \quad x_2^\star = 9$$



Geometry of Quadratic Programming

Quadratic programming:

$$\max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad x_1^2 + x_2^2$$
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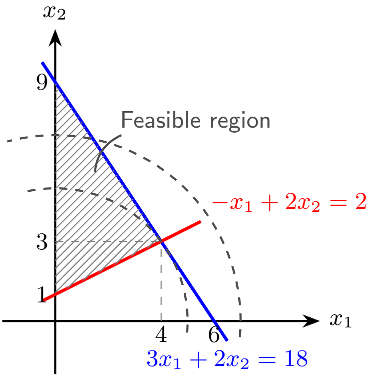

Geometry of Quadratic Programming

Quadratic programming:

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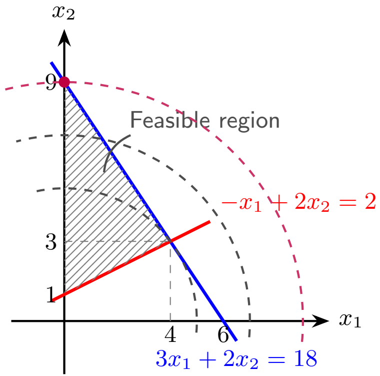
$$-x_1 + 2x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Optimal solution:

$$x_1^\star = 0 \quad x_2^\star = 9$$



Reformulating Quadratic Programming

- Reformulate the quadratic programming:

$$\begin{aligned} \max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \leq 18 \\ & -x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \leq 18 \\ & x_1 - 2x_2 \leq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Reformulating Quadratic Programming

- Reformulate the quadratic programming:

$$\begin{aligned} \max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \leq 18 \\ & -x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \leq 18 \\ & x_1 - 2x_2 \leq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Let $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 18 \\ -2 \end{bmatrix}$
- For decision variables $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, we have

$$\begin{aligned} \max_{\boldsymbol{x} \in \mathbb{R}^2} \quad & \|\boldsymbol{x}\|_2^2 \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

with the objective function $\|\boldsymbol{x}\|_2^2 = \boldsymbol{x}^\top \boldsymbol{x}$.

General Form of Quadratic Programming

- Quadratic programming:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \quad \underbrace{\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x}}_{\text{quadratic}} + \underbrace{\boldsymbol{c}^\top \boldsymbol{x}}_{\text{linear}}$$

$$\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

  - Symmetric matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$
  - $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$

- In the case of symmetric positive definite $\boldsymbol{Q}$, it is just a regression!

$$\|\boldsymbol{\varepsilon}\|_2^2 = \|\boldsymbol{b}_0 - \boldsymbol{A}_0\boldsymbol{x}\|_2^2 = \underbrace{\boldsymbol{x}^\top \boldsymbol{A}_0^\top \boldsymbol{A}_0\boldsymbol{x}}_{\text{quadratic}} - 2\underbrace{\boldsymbol{b}_0^\top \boldsymbol{A}_0\boldsymbol{x}}_{\text{linear}} + \underbrace{\boldsymbol{b}_0^\top \boldsymbol{b}_0}_{\text{constant}}$$

- What is the optimization of constrained linear regression?

Quick Summary

Linear programming (linear objective & constraints):

- **Linear** programming
- **Integer** programming (integer or binary variables)
- **Mixed-integer** programming (mix of continuous & integer variables)

Nonlinear programming (nonlinear objective & constraints):

- A special case: Quadratic programming
- How to reformulate ℓ_2-norm objective as quadratic programming?