Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 11: Ordinary Differential Equations

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Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

  "Class Participation Quiz 25"

  Time slot: **2:30PM – 3:00PM**

  on Canvas.

Definition

Ordinary Differential Equation (ODE) is an equation containing one or more derivatives of an unknown function with respect to a single independent variable.

- Differential equation, e.g.,

$$\frac{dy}{dx} = 2$$

- An ODE typically looks like:

$$\frac{dy}{dx} = f(x, y)$$

  - y: The dependent variable (what we want to find).
  - x: The independent variable.
  - $\frac{dy}{dx}$: The rate at which y changes over x.

ODE vs. PDE

Some ODEs:

- First-order ODE, e.g.,

$$\frac{dy}{dx} = e^x + y\sin(x)$$

- Second-order ODE, e.g.,

$$\underbrace{2\frac{d^2y}{dx^2}}_{\text{second-order}} + \underbrace{3\frac{dy}{dx}}_{\text{first-order}} + 2y = 0 \quad \Leftrightarrow \quad 2y'' + 3y' + 2y = 0 \quad \text{(notation with prime)}$$

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In contrast to ODEs, Partial Differential Equation (**PDE**) looks like:

- For a function $u(x, y, z)$ of three variables, Laplace's equation is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

- PDE involves unknown functions of multiple variables and their partial derivatives.

ODE Solution

How to find solution of differential equation?

- Example:

$$\frac{dy}{dx} = 2 \quad \Rightarrow \quad y = 2x + c \quad \text{for some constant } c$$

  - imposing initial value condition $y(0) = 2$, then $y = 2x + 2$

ODE Solution

How to find solution of ODEs?

- If we have an ODE of this form:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

  - **Separable**:

$$g(y)dy = f(x)dx$$

  - Integration:

$$\int g(y)dy = \int f(x)dx$$

- Example: $\dfrac{dP}{dt} = kP$
  - **Separable**:

$$\frac{1}{P}dP = k\,dt$$

  - Integration:

$$\int \frac{1}{P}dP = \int k\,dt \quad \Rightarrow \quad \ln|P| = kt + c \quad \Rightarrow P = c \cdot e^{kt}$$

ODE Solution

How to find solution of ODEs?

- If we have an ODE of this form:

$$M(x, y)dx + N(x, y)dy = 0$$

  and there exists a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N(x, y)$$

  The general solution to an exact equation is $\phi(x, y) = c$.

- Example: $(2x + y)dx + (x + 2y)dy = 0$
  - Assume

$$\frac{\partial \phi}{\partial x} = 2x + y \quad \Rightarrow \quad \phi(x, y) = x^2 + xy + g(y)$$

  - Then

$$\frac{\partial \phi}{\partial y} = x + g'(y) = x + 2y \quad \Rightarrow \quad g'(y) = 2y$$

$$\int g'(y)dy = \int 2y\, dy \quad \Rightarrow \quad g(y) = y^2 + c$$

  - Thus, we have $x^2 + xy + y^2 = c$ for some c

ODE Solution

How to find solution of ODEs?

- If we have an ODE of this form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

- The integrating factor (μ): We need to find $\mu(x)$ such that the left-hand side becomes a single derivative.

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x)$$

- Since

$$\frac{d}{dx}(\mu(x)y) = \mu'(x)y + \mu(x)y'$$

- Thus, we have

$$\mu'(x)y = \mu(x)p(x)y \quad \Rightarrow \quad \mu'(x) = \mu(x)p(x) \quad \Rightarrow \quad \mu(x) = e^{\int p(x)dx}$$

$$\mu(x)y = \int \mu(x)q(x)dx \quad \text{solve for} \quad y = \frac{1}{\mu(x)}\left(\int \mu(x)q(x)dx + c\right)$$

ODE Solution

Step-by-step analytical solution for the first-order linear ODE:

$$\frac{dy}{dx} + 2y = e^x$$

- Compare the example to the standard form $\dfrac{dy}{dx} + p(x)y = q(x)$:
  - $p(x) = 2$
  - $q(x) = e^x$
- The integrating factor is defined as $\mu(x) = e^{\int p(x)dx}$:

$$\mu(x) = e^{\int 2dx} = e^{2x}$$

- The exact equation is defined as

$$y = \frac{1}{\mu(x)} \left(\int \mu(x)q(x)dx + c \right)$$

- Thus, we have

$$y = \frac{1}{3}e^x + c \cdot e^{-2x}$$

ODE Solution

Step-by-step analytical solution for the first-order linear ODE:

$$\frac{dy}{dx} + \left(\frac{1}{x}\right) y = \sin(x)$$

- Compare the example to the standard form $\dfrac{dy}{dx} + p(x)y = q(x)$:
  - $p(x) = \dfrac{1}{x}$
  - $q(x) = \sin(x)$
- The formula for the integrating factor is $\mu(x) = e^{\int p(x)dx}$.

$$\mu(x) = e^{\int \frac{1}{x}dx} \quad \Rightarrow \quad \mu(x) = e^{\ln|x|} \quad \Rightarrow \quad \mu(x) = x$$

- The exact equation is defined as

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)q(x)dx + c\right)$$

- Thus, we have

$$x = \frac{1}{x}\int x\sin(x)dx = \frac{1}{x}(-x\cos(x) + \sin(x) + c)$$

General ODE

General form of an nth-order linear ODE:

- **Equation**: ODEs can be represented in a general linear form:

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = q(x)$$

- **Components**:
  - $y^{(n)}$ represents the nth derivative of y with respect to x.
  - $p_i(x)$ are the coefficient functions.
  - $q(x)$ is the non-homogeneous term.
- Initial points:

$$y(x_0) = b_0, \quad y'(x_0) = b_1, \quad \ldots, \quad y^{(n-1)}(x_0) = b_{n-1}$$

General ODE

Solve $y'' - y = e^{2x}$:

- Method of **Undetermined Coefficients**

$$y = \underbrace{y_h}_{\text{homogeneous solution}} + \underbrace{y_p}_{\text{particular solution}}$$

  - Solve the associated homogeneous equation:

$$y'' - y = 0$$

  - The **characteristic equation** is:

$$r^2 - 1 = 0 \implies (r-1)(r+1) = 0$$

  - The roots are $r_1 = 1$ and $r_2 = -1$.
  - Thus, the homogeneous solution is:

$$y_h = c_1 e^x + c_2 e^{-x}$$

General ODE

Solve $y'' - y = e^{2x}$:

- Method of **Undetermined Coefficients**

$$y = \underbrace{y_h}_{\text{homogeneous solution}} + \underbrace{y_p}_{\text{particular solution}}$$

  - Since the right-hand side is e^{2x}, we assume a particular solution of the form:
$$y_p = Ae^{2x}$$
  - Now, we find the derivatives: $y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$
  - Substitute these into the original differential equation ($y'' - y = e^{2x}$):
$$(4Ae^{2x}) - (Ae^{2x}) = e^{2x} \quad \Rightarrow \quad 3Ae^{2x} = e^{2x}$$
  - So, the particular solution is:
$$y_p = \frac{1}{3}e^{2x}$$

- Combining the two parts ($y = y_h + y_p$), we have:

$$y = c_1e^{x} + c_2e^{-x} + \frac{1}{3}e^{2x}$$

Concept of Linear Independence

- **Core Idea**: A set of functions $\{y_1, y_2, \ldots, y_n\}$ is linearly independent on an interval I if no function in the set can be expressed as a linear combination of the others.
- **Formal Definition**: The functions are linearly independent if the only constants $c_1, c_2, \ldots, c_n$ that satisfy:

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$

for all x in I are the trivial solutions: $c_1 = c_2 = \cdots = c_n = 0$.

Wronskian Determinant

To test for linear independence of n solutions to an nth-order linear homogeneous ODE, we construct the **Wronskian determinant**:

$$W(y_1, \ldots, y_n)(x) = \det \underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}}_{\text{Wronskian matrix}}$$

- **Test**: If there exists at least one point x_0 in the interval where $W \neq 0$, then the functions are linearly independent.
- **Significance**: If $W = 0$ everywhere on the interval, the solutions are linearly dependent.

Wronskian Determinant

Why it matters for ODEs?

- **Existence**: For an nth-order linear homogeneous equation, there always exists a set of n linearly independent solutions.
- **General Solution**: If $\{y_1, y_2, \ldots, y_n\}$ is a linearly independent set (a "Fundamental Set"), then every solution to the ODE can be written as:

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

- Example 1: $y_1 = e^x$, $y_2 = e^{2x}$

$$W(y_1, y_2)(x) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

- Example 2: $y_1 = \sin x, y_2 = \cos x$

$$W(y_1, y_2)(x) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} = -1$$

- Example 3: $y_1 = e^x, y_2 = e^{-x}$?

Wronskian Determinant

The solution to $y'' - y = e^{2x}$:

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{3} e^{2x}$$

- Wronskian determinant:

$$W(y_1, y_2, y_3)(x) = \det \begin{bmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{bmatrix}$$

Quick Summary

Monday's Class:

- Definition of ODE: Single independent variable, e.g., x
- PDE: Multiple variables, e.g., x, y, z
- First-order linear ODE:
  - Separable ODE: $\frac{dy}{dx} = \frac{f(x)}{g(y)}$
  - Non-separable ODE: $M(x, y)dx + N(x, y)dy = 0$
  - General form: $\frac{dy}{dx} + p(x)y = q(x)$
- nth-order linear ODE
- Linear independence with the Wronskian determinant