Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 1: Introduction to Applied Numerical Methods

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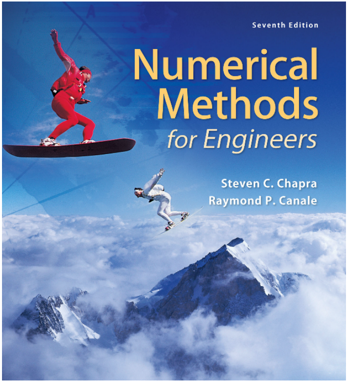
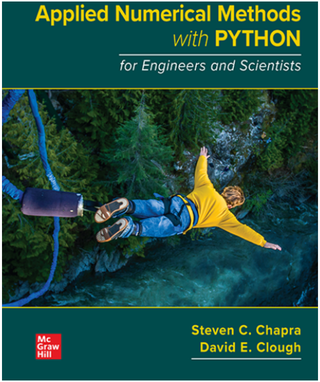
Course Description

- Numerical computing plays a critical role in solving complex problems in science and engineering.
- This course provides a comprehensive introduction to the numerical methods essential for civil and environmental engineering applications.
- Students will explore key topics such as linear algebra, differentiation and integration, nonlinear systems, and differential equations.
- Emphasis is placed on the practical, application-driven numerical solutions to common engineering challenges.

Reading Material





Course Structure

Week 1	Introduction to the Course & Logistics
Week 2	Mathematical Modeling & Engineering Problem Solving
Week 3	Introduction to Python Programming
Week 4	Introduction to Python Programming
Week 5	Modeling and Errors
Week 6	Modeling and Errors
Week 7	Nonlinear Equations
Week 8	Introduction to Applied Linear Algebra
Week 9	Introduction to Applied Linear Algebra
Week 10	Linear Algebraic Equations
Week 11	Ordinary Differential Equations
Week 12	Optimization Techniques
Week 13	Optimization Techniques
Week 14	Curve Fitting

Quick Summary

Monday's Class:

- Course structure

Thank you for attending this class!

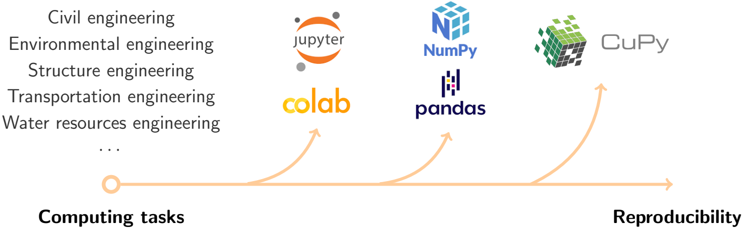
Coding Software

Python is an open-source programming language, supporting the development of numerical computing, artificial intelligence, data science, and etc.

- **Colab** platform: Google Colaboratory, or Google Colab for short, is a free, cloud-based Jupyter Notebook environment provided by Google. `https://colab.research.google.com`
- `NumPy` package: The fundamental package for scientific computing with Python. `https://numpy.org`
- `pandas` package: A fast, powerful, flexible and easy to use open source data analysis and manipulation tool, built on top of the Python programming language.

Coding Software

- The last mile of applied numerical methods for civil engineering?



- Recommendation for this course: Colab + NumPy

How to understand

Applied Numerical Methods for Civil Engineering?

Numerical methods are techniques by which **mathematical problems** are formulated so that they can be solved with **arithmetic operations**.

Study Motivation

Why you should study numerical methods?

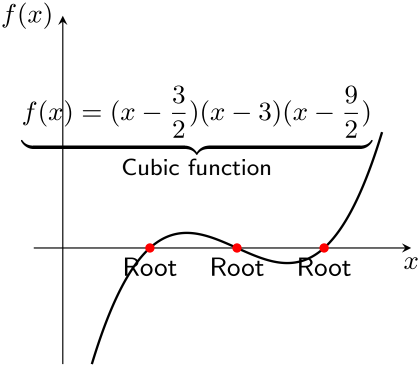
- Numerical methods are extremely powerful problem-solving tools. They are capable of handling large systems of equations, nonlinearities, and complicated geometries that are common in engineering practice and that are often impossible to solve analytically.
- Many engineering problems can be easily solved by numerical methods with computer programming.
- Numerical methods can help reinforce your understanding of mathematics because one function of numerical methods is to reduce higher mathematics to basic arithmetic operations.

Let's get started! **Basic Mathematical Background**.

❶ Roots of Equations

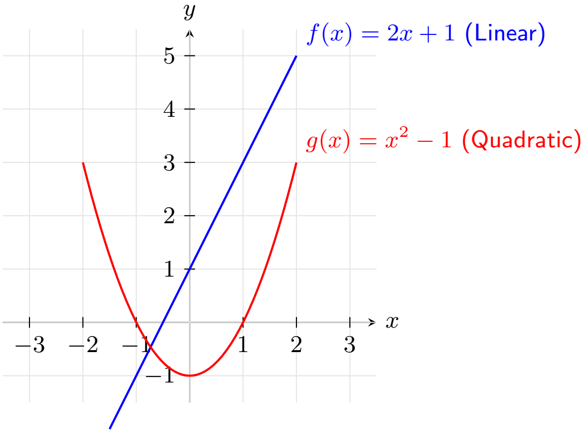
❶ Roots of Equations

- Solving $f(x) = 0$ for x.
- These problems are concerned with the value of a variable that satisfies a single nonlinear equation.
- These problems are especially valuable in engineering design where it is often impossible to explicitly solve design equations for variables.



❶ Roots of Equations

What is the difference between linear and nonlinear equations?



- Quadratic equation is a nonlinear equation.

❶ Roots of Equations

Study objectives today:

- Solve quadratic equations by factoring.
- Solve quadratic equations by the quadratic formula.

❶ Roots of Equations

- **Definition of Quadratic Equation**. An equation containing a second-degree polynomial is called a quadratic equation.
- Examples:

$$\underbrace{2x^2}_{\text{second-degree polynomial}} + 3x + 1 = 0 \qquad \underbrace{x^2}_{\text{second-degree polynomial}} - 4 = 0$$

- Standard form:

$$\underbrace{ax^2}_{\text{second-degree polynomial}} + bx + c = 0$$

  where a, b, and c are real numbers, and $a \neq 0$.
- They are used in countless ways in engineering practice.

❶ Roots of Equations

Zero-product property

- The zero-product property states that

  If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

  where a and b are real numbers or algebraic expressions.
- Example:

  $$(2x + 1)(x + 1) = 0$$

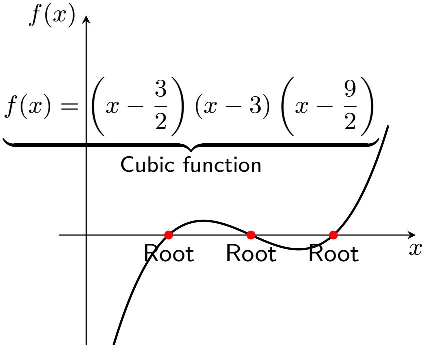
  The roots of this quadratic equation are $x = -\frac{1}{2}$ and $x = -1$.

❶ Roots of Equations

❶ Roots of Equations

- Solving $f(x) = 0$ for x.
- Zero-product property
- Roots of cubic equation:

$$x = \frac{3}{2}$$
$$x = 3$$
$$x = \frac{9}{2}$$



❶ Roots of Equations

Solve quadratic equations with **a leading coefficient of 1**: $x^2 + bx + c = 0$

- Find two numbers who sum equals b and whose product equals c, i.e.,

$$d + e = b \qquad d \cdot e = c$$

by using the zero-product property:

$$(x + d)(x + e) = x^2 + \underbrace{(d + e)}_{=b} x + \underbrace{d \cdot e}_{=c} = 0$$

Example. Roots of a simple quadratic equation.

Let's solve the quadratic equation $x^2 + x - 6 = 0$.

This is a quadratic equation with

$$d + e = b = 1 \qquad d \cdot e = c = -6$$

So we can factor the equation as:

$$(x - 2)(x + 3) = 0$$

As a result, we can find the solutions as $x = 2$ and $x = -3$.

❶ Roots of Equations

Example. Roots of a simple quadratic equation.

Let's solve the quadratic equation $x^2 - 5x + 6 = 0$.

This is standard quadratic equation of the form:

$$ax^2 + bx + c = 0$$

with

$$a = 1, \quad b = -5, \quad c = 6$$

So we can factor the equation as:

$$(x-2)(x-3) = 0$$

As a result, we can find the solutions as $x = 2$ and $x = 3$.

❶ Roots of Equations

Quadratic formula. Given $ax^2 + bx + c = 0$ $(a \neq 0)$, we can derive the quadratic formula by completing the square.

1. Move the constant term to the right-hand side:

$$ax^2 + bx = -c$$

2. Divide by a and let the leading coefficient be 1:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. Add $\frac{b^2}{4a^2}$ to both sides:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Use the square root property:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

❶ Roots of Equations

Quadratic formula. Given $ax^2 + bx + c = 0$ $(a \neq 0)$, the quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Python programming example.

```
import numpy as np

def quad_formula(a, b, c):
    term = np.sqrt(b**2 - 4*a*c)
    x1 = (-b + term) / (2*a)
    x2 = (-b - term) / (2*a)
    return x1, x2
```

Case study. Solve $9x^2 + 3x - 2 = (3x - 1)(3x + 2) = 0$.

```
a, b, c = 9, 3, -2
x1, x2 = quad_formula(a, b, c)
print(x1)
print(x2)
```

Basic Arithmetic Operations in Python

Python programming example.

```
1 a = 2
2 b = 3
3 print(a + b) # plus
4 print(a - b) # minus
5 print(a * b) # product
6 print(a / b) # division
7 print(a ** 2) # quadratic function
8 print(a ** 3) # cubic function
```

Corresponding **arithmetic operations**:

Line **3**: $a + b$

Line **4**: $a - b$

Line **5**: $a \cdot b$

Line **6**: $\frac{a}{b}$

Line **7**: a^2

Line **8**: a^3

Note: `a ** n` refers to a to the power of n.

Quick Summary

Wednesday's Class:

- Coding software (Python): No coding skill requirement
- Roots of equations (focus: quadratic equations)

Thank you for attending this class!

Mathematical Background

Study objectives today:

- ✓ ❶ Roots of equations
- ✗ → ✓ ❷ Linear algebraic equations
- ✗ → ✓ ❸ Optimization
- ✗ → ✓ ❹ Curve fitting
- ✗ → ✓ ❺ Integration
- ✗ → ✓ ❻ Ordinary differential equations

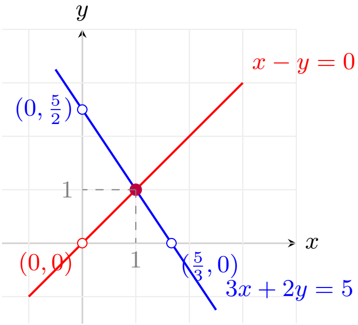
❷ Linear Algebraic Equations

What are linear algebraic equations?

- A **linear equation** is one where each term is either a constant or the product of a constant and a single variable. $\underbrace{2x}_{\text{product}} + \underbrace{1}_{\text{constant}} = 0$
- A **system of linear equations** is a set of multiple linear equations with the same variables.

Example with two variables x and y.

$$\begin{cases} 3x + 2y = 5 \\ x - y = 0 \end{cases}$$



❷ Linear Algebraic Equations

Solving by Hand: Substitution & Elimination

Example. Substitution Method.

Let's solve the following system of linear equations:

$$\begin{cases} 2x + y = 8 \\ x - y = 1 \end{cases}$$

1. From the second equation:

$$x = y + 1$$

2. Substitute into the first equation:

$$2(y + 1) + y = 8$$

3. Simplify:

$$2y + 2 + y = 8 \quad \Rightarrow \quad 3y = 6 \quad \Rightarrow \quad y = 2$$

4. Find x:

$$x = y + 1 = 3$$

❷ Linear Algebraic Equations

Solving by Hand: Substitution & Elimination

Example. Elimination Method.

Let's solve the following system of linear equations:

$$\begin{cases} 4x + 3y = 10 \\ 2x - y = 1 \end{cases}$$

1. Multiply second equation by 3:

$$6x - 3y = 3$$

2. Add to first equation:

$$(4x + 3y) + (6x - 3y) = 10 + 3$$

$$10x = 13 \quad \Rightarrow \quad x = 1.3$$

3. Substitute into $2x - y = 1$:

$$2 \times 1.3 - y = 1 \quad \Rightarrow \quad y = 2.6 - 1 = 1.6$$

❷ Linear Algebraic Equations

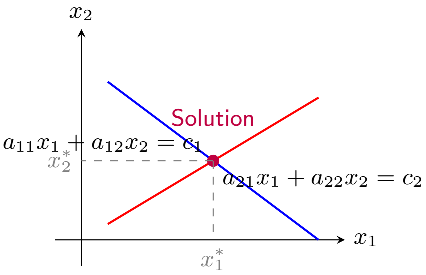
❷ Linear Algebraic Equations

- Solving

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = c_1 \\ a_{21}x_1 + a_{22}x_2 = c_2 \end{cases}$$

  for x_1 and x_2.

- Such equations arise in a variety of problem contexts and in all disciplines of engineering.
- They are also encountered in other areas of numerical methods such as curve fitting and differential equations.



❷ Linear Algebraic Equations

Example. Exact solution of a simple linear algebraic equation.

Let's solve the equation

$$\begin{cases} 4x_1 + x_2 = 1 & (1) \\ x_1 + 3x_2 = 2 & (2) \end{cases}$$

- Multiply Eq. (2) by 4 to align x_1 coefficients:

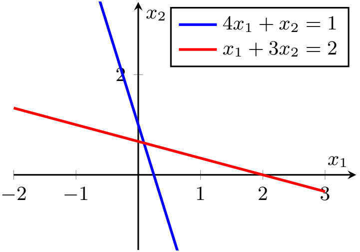
$$4x_1 + 12x_2 = 8 \qquad (3)$$

- Subtract Eq. (1) from Eq. (3):

$$(4x_1 + 12x_2) - (4x_1 + x_2) = 8 - 1$$

- Substitute $x_2 = \frac{7}{11}$ into Eq. (2):

$$x_1 = \frac{1}{11}$$



❷ Linear Algebraic Equations

Python programming example.

- Let's solve:

$$\begin{cases} 3x + 2y = 5 \\ x - y = 0 \end{cases}$$

- Try to solve by hand, and then check with Python.
- Define matrix

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$$

- Define vector

$$\boldsymbol{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

```python
import numpy as np

A = np.array([[3, 2], [1, -1]])
b = np.array([5, 0])
solution = np.linalg.solve(A, b)
print('Solution (x, y):', solution)
```

❷ Linear Algebraic Equations

Python programming example.

- Let's solve:

$$\begin{cases} x + y + z = 6 \\ 2y + 5z = -4 \\ 2x + 5y - z = 27 \end{cases}$$

- Try to solve by hand, and then check with Python.

```python
import numpy as np

A = np.array([[1, 1, 1], [0, 2, 5], [2, 5, -1]])
b = np.array([6, -4, 27])
solution = np.linalg.solve(A, b)
print('Solution (x, y, z):', solution)
```

❸ Optimization

Optimization is the process of finding the **best possible solution** to a problem, given certain constraints.

In **engineering**, we use optimization to:

- Minimize cost
- Maximize strength
- Minimize material usage
- Maximize efficiency
- Optimize traffic flow, energy usage, structural design, etc.

❸ Optimization

Optimization is the process of finding the **best possible solution** to a problem, given certain constraints.

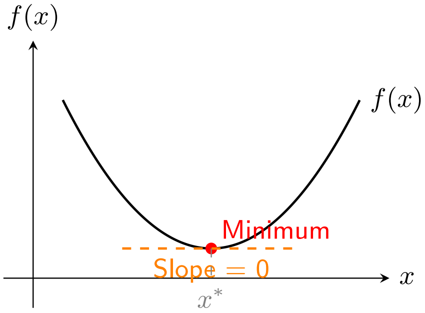
In **mathematics**, we look for the value of x that:

- Minimizes a function $f(x)$
- Maximizes a function $f(x)$

❸ Optimization

❸ Optimization

- Determine x that minimizes or maximizes $f(x)$.
- Such problems arise in a number of engineering design contexts and numerical methods.

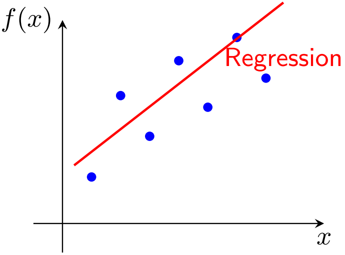


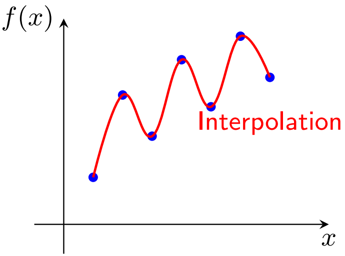
Example. Find x that minimizes $f(x) = x^2 - 4x + 5$.

❹ Curve Fitting

❹ Curve Fitting. The curve fitting techniques can be divided into two general categories:

- **Regression**: The strategy is to derive a single curve that represents the general trend of the data without necessarily matching any individual data points.
- **Interpolation**: The strategy is to fit a curve directly through the data points and use the curve to predict the intermediate values.





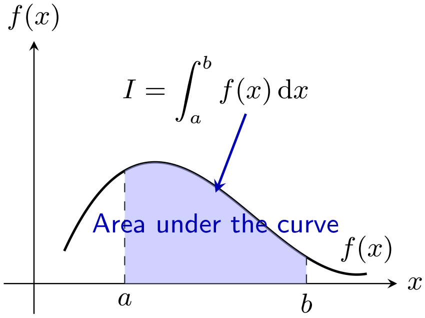
❺ Integration

❺ Integration is a mathematical tool for finding the **total accumulation** of a quantity over an interval.

- Determine the integration

$$I = \int_a^b f(x)\,\mathrm{d}x$$

- A physical interpretation of numerical integration is the determination of the area under a curve.
- Numerical integration formulas play an important role in the solution of differential equations.



❻ Ordinary Differential Equations

❻ Ordinary Differential Equations

- Ordinary differential equations are of great significance in engineering practice.

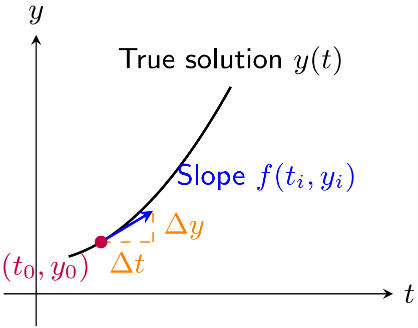
Given

$$\frac{\mathrm{d}y}{\mathrm{d}t} \cong \frac{\Delta y}{\Delta x} = f(t, y)$$

solve for y as a function of t.

$$y_{i+1} = y_i + f(t_i, y_i)\Delta t$$

with $\cong$ denoting "approximately equal to".



Quick Summary

Friday's Class:

- Linear algebraic equations
  - Theory: What linear systems are & how to solve by elimination/substitution.
  - Python: How to use `np.linalg.solve` to find solutions quickly.
- Optimization
- Curve fitting
- Integration
- Ordinary differential equations

Thank you for attending this class!