CGN 3405: Applied Numerical Methods for Civil Engineering

Topic: Nonlinear Equations

Submission Format: **PDF** Only

Q1: Derivation and Intuition (15 points)

The Newton-Raphson method can be derived using a first-order Taylor series expansion.

1. Given a nonlinear equation $f(x) = 0$, show the step-by-step derivation of the Newton-Raphson iteration formula starting from the Taylor series expansion around a current estimate x_n. **(5 points)**

2. Using the definition of a slope (slope $= \frac{\Delta y}{\Delta x}$), show how the tangent line crossing the x-axis at $(x_{n+1}, 0)$ leads to the Newton-Raphson formula. **(5 points)**

3. Explain the "Correction Term" $\frac{f(x_n)}{f'(x_n)}$. What happens to the step size if the function value $f(x_n)$ is very large versus when the slope $f'(x_n)$ is very steep? **(5 points)**

Q2: Manual Iteration (15 points)

Consider the function $f(x) = x^3 - 20 = 0$ with an initial guess of $x_0 = 3$.

1. Perform two iterations of the Newton-Raphson method to find x_2. Show all your work, including the calculation of the derivative $f'(x)$. **(7.5 points)**

2. Calculate the relative error (ϵ) for the second iteration (x_2) using the formula:

$$\epsilon = \frac{|x_{n+1} - x_n|}{|x_{n+1}|}$$

   Express your answer as a percentage. **(7.5 points)**

Q3: Convergence Analysis (10 points)

Newton-Raphson is known for its quadratic convergence under ideal conditions.

1. If the residual at iteration n is $\varepsilon_n = 0.1$ and the method is converging quadratically (assume $\epsilon_{n+1} = \varepsilon_n^2$), calculate the residuals for the next three iterations ($n = 1, 2, 3$). What does this suggest about the number of significant digits in each step? **(6 points)**

2. Identify one specific mathematical case where the Newton-Raphson method fails to achieve quadratic convergence and instead converges linearly. Provide the function example mentioned in the slides. **(4 points)**

Q4: Identifying the Order of Convergence (15 points)

In numerical analysis, we categorize sequences by their "Order of Convergence." For each of the following sequences converging to $x_\infty = 2$, use the limit definitions from the course to determine if the convergence is linear, superlinear, or quadratic. **(5 points for each)**

1. Sequence A (Linear): $x_n = 2 + (0.8)^n$.

   Hint: Show that $\frac{|x_{n+1}-2|}{|x_n-2|} \leq r$ where $r \in (0, 1)$.

2. Sequence B (Superlinear): $x_n = 2 + \frac{1}{n!}$.

   Hint: Prove that $\lim_{n\to\infty} \frac{|x_{n+1}-2|}{|x_n-2|} = 0$.

3. Sequence C (Quadratic): $x_n = 2 + (0.5)^{2^n}$.

   Hint: Verify that $\frac{|x_{n+1}-2|}{|x_n-2|^2} \leq M$ for some constant M.